

## A HAAGERUP INEQUALITY FOR $\tilde{A}_1 \times \tilde{A}_1$ AND $\tilde{A}_2$ BUILDINGS

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### Abstract

Haagerup's inequality for convolvers on free groups may be interpreted as a result on  $\tilde{A}_1$  buildings, i.e. trees. Here are proved analogous inequalities for discrete groups acting freely on the vertices of  $\tilde{A}_1 \times \tilde{A}_1$  and  $\tilde{A}_2$  buildings. The results apply in particular to groups of type-rotating automorphisms acting simply transitively on the vertices of such buildings. These results provide the first examples of higher rank groups with property (RD).

### 1 Introduction

U. Haagerup has given a beautiful and useful estimate for convolvers on a free group [H, Lemma 1.4]. Suppose  $\Gamma$  is the free group on a set of generators  $N_+$  and let  $N$  consist of the generators from  $N_+$  and their inverses. Each  $c \in \Gamma$  can be written uniquely as  $c = a_1 a_2 \cdots a_n$  with  $a_j \in N$  and  $a_j a_{j+1} \neq 1$ . This product is called the reduced word for  $c$ , and if the reduced word has  $n$  factors, we say that  $c$  is a word of length  $n$ . Haagerup's inequality applies to a function  $g \in \ell^1(\Gamma)$  supported on the words of length  $n$  and it asserts that

$$\|f * g\|_2 \leq (n + 1) \|f\|_2 \|g\|_2.$$

Denoting by  $\rho$  the right regular representation of  $\ell^1(\Gamma)$  on  $\ell^2(\Gamma)$ , the inequality reads  $\|\rho(g)\| \leq (n + 1) \|g\|_2$ .

In [H] Haagerup's inequality was used in the course of establishing that the reduced  $C^*$ -algebra of a free group on finitely many generators has the metric approximation property. In another development, it follows from Haagerup's inequality that Jolissaint's property (RD) holds for free groups.

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Haagerup’s inequality was extended to word hyperbolic groups in [J], [Ha1], proving that they too satisfy property (RD). This is an ingredient in the proof of the Novikov conjecture for word hyperbolic groups [CoMo]. An exposition of this last result may be found in [Co, Chapter III.5]. Also a recent paper of A. Nevo [Ne] provides a new application of the Haagerup inequality to ergodic theorems on groups. Another new application of the Haagerup inequality (to operator growth functions on groups) can be found in the recent paper by R. Grigorchuk and T. Nagnibeda [GN]. Actually in [GN], the main result of the present paper is wrongly attributed to [CMSt], as the referee of this paper pointed out.

Restating Haagerup’s proof geometrically, we find that it gives a result somewhat more general than was originally stated. Suppose  $\Gamma$  acts freely on the vertices of a tree, and let  $d(u, v)$  denote graph theoretic distance. Fix a vertex  $v_0$  and define the length of  $c \in \Gamma$  as

$$|c| = d(v_0, cv_0).$$

If  $g \in \ell^1(\Gamma)$  is supported on elements of length  $n$ , then  $\|\rho(g)\| \leq (n+1)\|g\|_2$ . To recover the original inequality, consider the free group  $\Gamma$  acting simply transitively on its Cayley graph with respect to the generating set  $N$  and take  $v_0$  to be  $1 \in \Gamma$ .

Trees are  $\tilde{A}_1$  buildings. This paper generalizes Haagerup’s inequality to  $\tilde{A}_1 \times \tilde{A}_1$  and  $\tilde{A}_2$  buildings. This is the first such generalization to “higher rank” groups of either Haagerup’s inequality or of property (RD).

Let  $\Delta$  be an  $\tilde{A}_1 \times \tilde{A}_1$  or  $\tilde{A}_2$  building. In §1.2 we define the *shape*,  $\sigma(u, v) \in \mathbb{N} \times \mathbb{N}$  between any two vertices  $u, v \in \Delta$  which essentially gives the dimensions of the convex hull of the two vertices in the building sense. The index of

$$\text{Aut}_S(\Delta) = \{c \in \text{Aut}(\Delta) \mid \sigma(cu, cv) = \sigma(u, v) \text{ for all } u, v \in \mathcal{V}_\Delta\}$$

in  $\text{Aut}(\Delta)$  is 1 or 2.

Define a function in two variables

$$p(m, n) = \begin{cases} (m+1)(n+1) & (\tilde{A}_1 \times \tilde{A}_1 \text{ case}) \\ 1/2(m+1)(n+1)(m+n+2)\sqrt{\max(m, n)+1} & (\tilde{A}_2 \text{ case}). \end{cases}$$

Our main result is the following.

**Theorem 1.1.** *Suppose  $\Delta$  is an  $\tilde{A}_1 \times \tilde{A}_1$  or  $\tilde{A}_2$  building and  $\Gamma \leq \text{Aut}_S(\Delta)$  acts freely on the vertices of  $\Delta$ . Fix any vertex  $v_0 \in \Delta$  and define a shape function  $\sigma$  on  $\Gamma$  by*

$$\sigma(c) = \sigma(v_0, cv_0).$$

If  $g \in \mathbb{C}\Gamma$  is supported on words of shape  $(m, n)$  and  $f \in \ell^2(\Gamma)$ , then

$$\|f * g\|_2 \leq p(m, n) \|f\|_2 \|g\|_2.$$

In the  $\tilde{A}_1 \times \tilde{A}_1$  case our bound is optimal. From an analysis of radial functions, we conjecture that the optimal value of  $p(m, n)$  in the  $\tilde{A}_2$  case is  $1/2(m+1)(n+1)(m+n+2)$ . Using a result in [Sw], A. Valette [V] has shown that there is a Haagerup inequality applying to radial functions on an  $\tilde{A}_n$  group.

Among buildings of dimension 2 only  $\tilde{A}_1 \times \tilde{A}_1$  and  $\tilde{A}_2$  buildings ever admit simply transitive actions on their vertices. It was this that led us to consider these two cases — by happy coincidence they are also the easiest. Indeed, Haagerup's original method, broadly conceived, handles  $\tilde{A}_1 \times \tilde{A}_1$  buildings, and only one new ingredient is required for the  $\tilde{A}_2$  case. It is natural to conjecture that analogous versions of Haagerup's inequality hold for all types of affine buildings. In fact, Valette has conjectured [FRaRos, p. 70] that any group acting properly and cocompactly on either a Riemannian symmetric space or an affine building has property (RD).

**1.1 Some analytical results used in the proofs.** We note the following well-known results which will be used on several occasions. Their proofs are straightforward and so are omitted.

LEMMA 1.2. *Let  $T : \mathcal{H} \rightarrow \mathcal{K}$  be an operator between two Hilbert spaces, and suppose we have orthogonal decompositions  $\mathcal{H} = \bigoplus_j \mathcal{H}_j$  and  $\mathcal{K} = \bigoplus_k \mathcal{K}_k$ . Expressing  $T$  as an operator matrix  $[T_{kj}]$  where  $T_{kj} : \mathcal{H}_j \rightarrow \mathcal{K}_k$ , one has*

$$\|T\| \leq \|[\|T_{kj}\|\|]\| \leq \left( \sum_{j,k} \|T_{kj}\|^2 \right)^{1/2}$$

where all norms are operator norms and  $[\|T_{kj}\|\|]$  is the matrix with scalar entries  $\|T_{kj}\|$ .

LEMMA 1.3. *If  $T = [t_{ij}]$ ,  $S = [s_{ij}]$  with  $0 \leq t_{ij} \leq s_{ij}$  for all  $i, j$ , then  $\|T\| \leq \|S\|$ .*

**1.2 The buildings and the shape between a pair of vertices.**

Given an  $\tilde{A}_n$  building  $\Delta$ , there is a type map  $\tau$  defined on the vertices of  $\Delta$  such that  $\tau(v) \in \mathbb{Z}/(n+1)\mathbb{Z}$  for each vertex  $v \in \Delta$ . A brief inductive argument shows that every  $a \in \text{Aut}(\Delta)$  gives rise to a permutation of the set of types. An automorphism  $a$  of  $\Delta$  is said to be *type-rotating* if there exists  $i \in \{0, 1, \dots, n\}$  such that  $\tau(av) = \tau(v) + i$  for all vertices  $v \in \Delta$ . More generally, suppose that  $\Delta$  is an  $\tilde{A}_{n_1} \times \dots \times \tilde{A}_{n_k}$  building. There is a

type map  $\tau$  on the vertices of  $\Delta$  where

$$\tau(v) = (\tau(v)_1, \dots, \tau(v)_k) \in \mathbb{Z}/(n_1 + 1)\mathbb{Z} \times \dots \times \mathbb{Z}/(n_k + 1)\mathbb{Z}$$

is a  $k$ -tuple. Again, any automorphism  $a$  of  $\Delta$  gives rise to a permutation on the set of types. We say such an  $a$  is *type-rotating* if there exists a  $k$ -tuple  $(i_1, \dots, i_k) \in \{0, 1, \dots, n_1\} \times \dots \times \{0, 1, \dots, n_k\}$  such that

$$\tau(av) = (\tau(v)_1 + i_1, \dots, \tau(v)_k + i_k).$$

It follows (see §4.1) that  $a$  is a type-rotating automorphism of  $\Delta$  if and only if it is a Cartesian product of type-rotating automorphisms of the  $k$  factors. Recall that an  $\tilde{A}_1$  building is a tree and note that any automorphism of an  $\tilde{A}_1$  building is type-rotating.

Henceforth, let  $\Delta$  be an  $\tilde{A}_1 \times \tilde{A}_1$  or  $\tilde{A}_2$  building. The subgroup of type-rotating automorphisms will then be of index at most 2 in  $\text{Aut}(\Delta)$ . This follows from the facts that  $[D_8 : C_2 \times C_2] = [D_6 : C_3] = 2$ , where  $C_n$  is the cyclic group and  $D_n$  is the dihedral group of order  $n$ . An *apartment* in  $\Delta$  is a chamber subcomplex of  $\Delta$  isomorphic to the corresponding Coxeter complex. Thus an apartment in  $\Delta$  is a plane tessellated by squares in the  $\tilde{A}_1 \times \tilde{A}_1$  case and by equilateral triangles in the  $\tilde{A}_2$  case.

Denote by  $\mathcal{V}_\Delta$  the vertex set of  $\Delta$ . Any two vertices  $u, v \in \mathcal{V}_\Delta$  belong to a common apartment. The convex hull, in the sense of buildings, between two vertices  $u$  and  $v$  is depicted in Figure 1, with various degeneracies possible. Fundamental properties of buildings imply that any apartment containing  $u$  and  $v$  must also contain their convex hull.

Define the *distance*,  $d(u, v)$ , between  $u$  and  $v$  to be the graph theoretic distance on the one-skeleton of  $\Delta$ . Any path from  $u$  to  $v$  of length  $d(u, v)$  lies in their convex hull, and the union of the vertices in such paths is exactly the set of vertices in the convex hull. Note that although this last statement is true for  $\tilde{A}_1 \times \tilde{A}_1$  and  $\tilde{A}_2$  buildings, it does not hold for arbitrary affine buildings. For example in a  $\tilde{G}_2$  building it fails for two vertices whose convex hull contains at least three chambers.

We define the *shape*  $\sigma(u, v)$  of the ordered pair of vertices  $(u, v) \in \mathcal{V}_\Delta \times \mathcal{V}_\Delta$  to be the pair  $(m, n) \in \mathbb{N} \times \mathbb{N}$  as indicated in Figure 1. Note that  $d(u, v) = m + n$ . In the  $\tilde{A}_2$  case the arrows in Figure 1 point in the direction of cyclically increasing type, i.e.  $\{\dots, 0, 1, 2, 0, 1, \dots\}$ . In the  $\tilde{A}_1 \times \tilde{A}_1$  case the components of  $\sigma(u, v)$  indicate the relative contributions to  $d(u, v)$  from the two  $\tilde{A}_1$  factors. More specifically, suppose  $\Delta = \Delta_1 \times \Delta_2$  where  $\Delta_1$  and  $\Delta_2$  are  $\tilde{A}_1$  buildings, i.e. trees. If  $v = (v_1, v_2)$  and  $w = (w_1, w_2)$  are

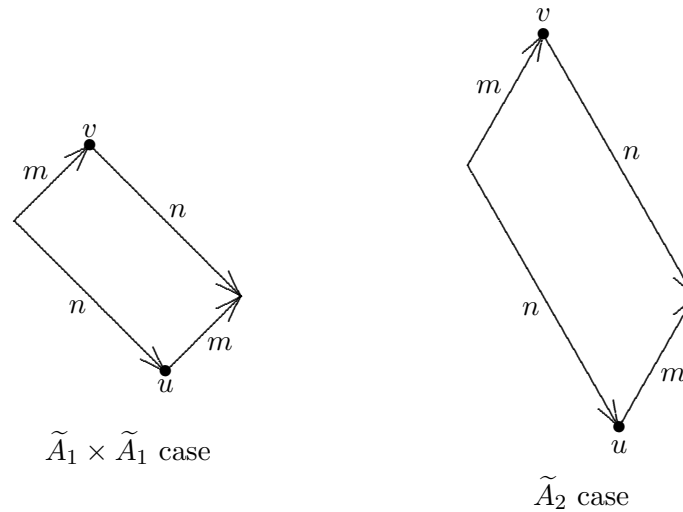


Figure 1: Convex hull of two vertices.

vertices of  $\Delta$ , the *shape* from  $v$  to  $w$  is

$$\sigma(v, w) = (d(v_1, w_1), d(v_2, w_2))$$

where  $d$  denotes the usual graph-theoretic distance on a tree. An edge in  $\Delta$  connects the vertices  $v$  and  $w$  if  $\sigma(v, w) = (0, 1)$  or  $\sigma(v, w) = (1, 0)$ .

LEMMA 1.4. *Suppose  $m_1, m_2, n_1, n_2 \in \mathbb{N}$  and  $\sigma(u, w) = (m_1 + m_2, n_1 + n_2)$  for vertices  $u, w \in \mathcal{V}_\Delta$ . Then there is a unique vertex  $v \in \mathcal{V}_\Delta$  such that*

$$\sigma(u, v) = (m_1, n_1) \quad \text{and} \quad \sigma(v, w) = (m_2, n_2).$$

*Proof.* Such a  $v \in \mathcal{V}_\Delta$  satisfies  $d(u, w) = d(u, v) + d(v, w)$  so it must lie in the convex hull of  $u$  and  $w$ . Inside the convex hull existence and uniqueness of  $v$  are clear.  $\square$

It is a direct consequence of the definitions that every type-rotating automorphism  $a \in \text{Aut}(\Delta)$  preserves shape in the sense that  $\sigma(au, av) = \sigma(u, v)$  for all  $u, v \in \mathcal{V}_\Delta$ . Furthermore, if  $b \in \text{Aut}(\Delta)$  is not type-rotating then  $\sigma(bu, bv) = (n, m)$  whenever  $\sigma(u, v) = (m, n)$ . Thus an element  $a \in \text{Aut}(\Delta)$  preserves shape if and only if it is type-rotating. Thus the group  $\text{Aut}_\mathcal{S}(\Delta)$  of shape-preserving automorphisms coincides with the group of type-rotating automorphisms.

We refer to section 4 for a more detailed analysis of the interesting special case where the group  $\Gamma \leq \text{Aut}_S(\Delta)$  acts simply transitively on  $\mathcal{V}_\Delta$ .

**1.3 Free actions and groupoids.** Suppose now that  $\Gamma \leq \text{Aut}_S(\Delta)$  acts freely, but not necessarily transitively, on the vertices  $\mathcal{V}_\Delta$  of  $\Delta$ . This induces an action of  $\Gamma$  on  $\mathcal{V}_\Delta \times \mathcal{V}_\Delta$ . Denote by  $[u, v]$  the orbit of the ordered pair  $(u, v) \in \mathcal{V}_\Delta \times \mathcal{V}_\Delta$  under the left action of  $\Gamma$ . Define

$$\Gamma' = \Gamma \backslash (\mathcal{V}_\Delta \times \mathcal{V}_\Delta),$$

the set of  $\Gamma$ -orbits of  $\mathcal{V}_\Delta \times \mathcal{V}_\Delta$ . Given  $[u, v]$  and  $[v, w] \in \Gamma'$ , define their product as

$$[u, v][v, w] = [u, w].$$

The product of  $[u, v]$  and  $[x, y]$  is not always defined. It exists only if  $v$  and  $x$  lie in the same  $\Gamma$ -orbit, and in that case it is well defined because  $\Gamma$  acts freely on  $\mathcal{V}_\Delta$ . Specifically, if  $bx = v$  for  $b \in \Gamma$ , then

$$[u, v][x, y] = [u, v][bx, by] = [u, by].$$

The set  $\Gamma'$  with this product satisfies the axioms of a groupoid, see [Co, Chapter II.5]:

DEFINITION 1.5. A *groupoid* consists of a set  $\mathcal{G}$ , a distinguished subset  $\mathcal{G}^{(0)} \subset \mathcal{G}$ , two maps  $s, r : \mathcal{G} \rightarrow \mathcal{G}^{(0)}$  and a law of composition  $(\alpha_1, \alpha_2) \in \mathcal{G}^{(2)} \mapsto \alpha_1\alpha_2 \in \mathcal{G}$ , with domain  $\mathcal{G}^{(2)} = \{(\alpha_1, \alpha_2) \in \mathcal{G} \times \mathcal{G} : s(\alpha_1) = r(\alpha_2)\}$ , such that

- $s(\alpha_1\alpha_2) = s(\alpha_2)$ ,  $r(\alpha_1\alpha_2) = r(\alpha_1)$ , for all  $(\alpha_1, \alpha_2) \in \mathcal{G}^{(2)}$ ;
- $s(\alpha) = r(\alpha) = \alpha$ , for all  $\alpha \in \mathcal{G}^{(0)}$ ;
- $\alpha s(\alpha) = \alpha$ ,  $r(\alpha)\alpha = \alpha$ , for all  $\alpha \in \mathcal{G}$ ;
- $(\alpha_1\alpha_2)\alpha_3 = \alpha_1(\alpha_2\alpha_3)$ ;
- each  $\alpha$  has a two-sided inverse  $\alpha^{-1}$ , with  $\alpha\alpha^{-1} = r(\alpha)$ ,  $\alpha^{-1}\alpha = s(\alpha)$ .

The maps  $r$  and  $s$  are called the *range* and *source* maps. An element  $\iota \in \mathcal{G}^{(0)}$  is called a *unit*. The units in  $\Gamma'$  are of the form  $[v, v]$  for  $v \in \mathcal{V}_\Delta$ . If  $\alpha = [u, v] \in \Gamma'$  then  $r(\alpha) = [u, u]$ ,  $s(\alpha) = [v, v]$  and  $\alpha^{-1} = [v, u]$ .

DEFINITION 1.6. Let  $\mathcal{G}$  be a groupoid and  $X$  be a set together with a surjection  $s' : X \rightarrow \mathcal{G}^{(0)}$ . Form the fibred product

$$X * \mathcal{G} = \{(v, \alpha) \in X \times \mathcal{G} \mid s'(v) = r(\alpha)\}$$

A (right) *groupoid action* of  $\mathcal{G}$  on  $X$  is a map  $(v, \alpha) \mapsto v\alpha \in X$  from  $X * \mathcal{G}$  to  $X$  such that

- $s'(v\alpha) = s(\alpha)$ ;
- $v(\alpha\beta) = (v\alpha)\beta$ , whenever  $(v, \alpha) \in X * \mathcal{G}$  and  $(\alpha, \beta) \in \mathcal{G}^{(2)}$ ;

- $vs'(v) = v$  for all  $v \in X$ .

We think of  $s'$  as being a ‘generalized source’ map for the action. A groupoid action is called *simply transitive* if in addition

- given  $v, w \in X$  there exists a unique  $\alpha \in \mathcal{G}$  such that  $(v, \alpha) \in X * \mathcal{G}$  and  $v\alpha = w$ .

We refer the reader to [Re] and [MuW, Section 2] for details on groupoid actions. When  $\mathcal{G}$  acts simply transitively on  $X$  and  $v, w \in X$ , we write  $v^{-1}w$  for the unique  $\alpha \in \mathcal{G}$  such that  $v\alpha = w$ . This notation is possibly misleading since  $v^{-1}$  has no independent existence. However, if  $(v, \beta), (w, \gamma) \in X * \mathcal{G}$  then

$$(v\beta)^{-1}(w\gamma) = \beta^{-1}(v^{-1}w)\gamma$$

as is easily checked.

We define a simply transitive groupoid action of  $\Gamma'$  on  $\mathcal{V}_\Delta$  via

$$v[v, w] = w.$$

Thus  $s'(v) = [v, v]$  and  $(u, [v, w]) \in X * \mathcal{G}$  exactly when  $u$  and  $v$  lie in the same  $\Gamma$ -orbit. If  $v, w \in \mathcal{V}_\Delta$  then  $v\alpha = w$  if and only if  $\alpha = [v, w]$  so that  $v^{-1}w = [v, w]$ .

DEFINITION 1.7. Let  $\Delta$  be an  $\tilde{A}_1 \times \tilde{A}_1$  or  $\tilde{A}_2$  building,  $\mathcal{V}_\Delta$  its vertex set and  $\sigma_\Delta$  its shape function. Let  $\mathcal{G}$  be a groupoid acting on  $\mathcal{V}_\Delta$  on the right. A *shape function* on  $\mathcal{G}$  compatible with the action is a map  $\sigma_\mathcal{G} : \mathcal{G} \rightarrow \mathbb{N} \times \mathbb{N}$  such that

$$\sigma_\Delta(v, v\alpha) = \sigma_\mathcal{G}(\alpha)$$

whenever  $v \in \mathcal{V}_\Delta, \alpha \in \mathcal{G}$  and  $v\alpha$  is defined.

Henceforth we will omit the subscripts and denote all shape functions by  $\sigma$ . A shape function on  $\Gamma'$  compatible with the action of  $\Gamma'$  on  $\Delta$  is given by

$$\sigma([u, v]) = \sigma(u, v)$$

for any  $u, v \in \mathcal{V}_\Delta$ . This is well-defined since  $\Gamma \leq \text{Aut}_S(\Delta)$  acts on  $\Delta$  by shape-preserving automorphisms.

LEMMA 1.8. Let  $\Delta$  be an  $\tilde{A}_1 \times \tilde{A}_1$  or  $\tilde{A}_2$  building. Let  $\mathcal{G}$  be a groupoid endowed with a simply transitive action on  $\mathcal{V}_\Delta$  and a compatible shape function. Let  $m_1, m_2, n_1, n_2 \in \mathbb{N}$ . Whenever  $\alpha \in \mathcal{G}$  has shape  $(m_1 + m_2, n_1 + n_2)$  there exist unique  $\beta, \gamma \in \mathcal{G}$  such that

$$\alpha = \beta\gamma, \quad \sigma(\beta) = (m_1, n_1) \quad \text{and} \quad \sigma(\gamma) = (m_2, n_2).$$

*Proof.* Fix  $u \in \mathcal{V}_\Delta$  such that  $w = u\alpha$  exists. A pair  $(\beta, \gamma)$  satisfies the required conditions if and only if

$$\sigma(\beta) = (m_1, n_1), \quad \sigma(\beta^{-1}\alpha) = (m_2, n_2) \quad \text{and} \quad \gamma = \beta^{-1}\alpha.$$

This happens if and only if  $v = u\beta$  satisfies

$$\sigma(u, v) = (m_1, n_1) \quad \text{and} \quad \sigma(v, w) = (m_2, n_2).$$

By Lemma 1.4, there is exactly one such vertex  $v$ . Hence the pair  $(\beta, \gamma) = (u^{-1}v, v^{-1}w)$  uniquely satisfies the required conditions.  $\square$

To summarize, any group  $\Gamma \leq \text{Aut}_S(\Delta)$  with a free left action on  $\mathcal{V}_\Delta$  gives rise to a groupoid  $\Gamma'$  with a simply transitive (right) groupoid action on  $\mathcal{V}_\Delta$  and a compatible shape function. The actions of  $\Gamma$  and  $\Gamma'$  commute in the sense that

$$c(v\alpha) = (cv)\alpha$$

for  $c \in \Gamma$ ,  $v \in \mathcal{V}_\Delta$  and  $\alpha \in \Gamma'$  provided one side or the other exist. We will call  $\Gamma'$  the *commutant groupoid* of  $\Gamma$ . Note that the left action of  $\Gamma$  on  $\mathcal{V}_\Delta$  preserves the building structure and that the right action of  $\Gamma'$  on  $\mathcal{V}_\Delta$  does not.

If, as in §4,  $\Gamma$  acts simply transitively on  $\mathcal{V}_\Delta$ , then  $\Gamma'$  is a group isomorphic to  $\Gamma$  and the groupoid action of  $\Gamma'$  on  $\mathcal{V}_\Delta$  constructed in the above manner is equivalent to a right group action of  $\Gamma$  on  $\mathcal{V}_\Delta$ . For any fixed  $v_0 \in \mathcal{V}_\Delta$ , the map  $c \mapsto [v_0, cv_0]$  gives an isomorphism  $\Gamma \rightarrow \Gamma'$ . This isomorphism depends on the choice of  $v_0$ , varying up to inner automorphisms of  $\Gamma$ . Our results will be phrased in the language of groupoid actions with implications for free actions of groups on  $\mathcal{V}_\Delta$  via the above construction.

**1.4 Convolutions.** If  $\mathcal{G}$  is a groupoid, let  $\mathbb{C}\mathcal{G}$  denote the space of finitely supported complex valued functions on  $\mathcal{G}$ . The convolution product on  $\mathbb{C}\mathcal{G}$  defined by

$$(g_1 * g_2)(\gamma) = \sum_{\alpha\beta=\gamma} g_1(\alpha)g_2(\beta)$$

makes  $\mathbb{C}\mathcal{G}$  into an associative algebra. Suppose that  $\mathcal{G}$  acts simply transitively on a set  $X$ . For  $g \in \mathbb{C}\mathcal{G}$ , define  $\rho(g): \ell^2(X) \rightarrow \ell^2(X)$  by

$$(\rho(g)f)(v) = (f * g)(v) = \sum_{\substack{u \in X \\ u\alpha=v}} f(u)g(\alpha).$$

It follows from the simple transitivity of the action of  $\mathcal{G}$  that  $\rho(g)f$  actually lies in  $\ell^2(X)$ ; indeed that  $\|f * g\|_2 \leq \|f\|_2 \|g\|_1$ . The map  $\rho$  defines a right action of  $\mathbb{C}\mathcal{G}$  on  $\ell^2(X)$ .



Let  $\Delta$  be an  $\tilde{A}_1 \times \tilde{A}_1$  or  $\tilde{A}_2$  building and let

$$p(m, n) = \begin{cases} (m+1)(n+1) & (\tilde{A}_1 \times \tilde{A}_1 \text{ case}) \\ 1/2(m+1)(n+1)(m+n+2)\sqrt{\max(m, n)+1} & (\tilde{A}_2 \text{ case}). \end{cases} \quad (1)$$

In sections 2 and 3 we prove the following result:

**Theorem 1.9.** *Suppose  $\mathcal{G}$  is a groupoid acting simply transitively on  $\mathcal{V}_\Delta$  and  $\sigma: \mathcal{G} \rightarrow \mathbb{N}$  is a shape function compatible with the action. Fix any  $(m, n) \in \mathbb{N} \times \mathbb{N}$ . If  $g \in \mathbb{C}\mathcal{G}$  is supported on elements of shape  $(m, n)$ , then  $\|\rho(g)\| \leq p(m, n)\|g\|_2$ .*

Equivalently, the conclusion asserts that

$$\|f * g\|_2 \leq p(m, n)\|f\|_2\|g\|_2.$$

We can and will restrict to nonnegative and finitely supported  $f$  and  $g$  when proving this.

If  $\Gamma$  is a discrete group, we define convolution between  $f \in \ell^2(\Gamma)$  and  $g \in \mathbb{C}\Gamma$  in the usual way:

$$(f * g)(c) = \sum_{ab=c} f(a)g(b).$$

Theorem 1.1 is a consequence of Theorem 1.9.

*Proof of Theorem 1.1.* Let  $\Gamma'$  be the commutant groupoid of  $\Gamma$ , as defined in §1.3. Given  $f \in \ell^2(\Gamma)$  and  $g \in \mathbb{C}\Gamma$ , define  $f' \in \ell^2(\mathcal{V}_\Delta)$  and  $g' \in \mathbb{C}\Gamma'$  by

$$\begin{aligned} f'(cv_0) &= f(c) && \text{for } c \in \Gamma \\ f'(w) &= 0 && \text{unless } w \in \Gamma v_0 \\ g'([dv_0, cv_0]) &= g(d^{-1}c) && \text{for } c, d \in \Gamma \\ g'([w_1, w_2]) &= 0 && \text{unless } w_1, w_2 \in \Gamma v_0. \end{aligned}$$

It is then immediate that  $f' * g'$  is related to  $f * g$  via

$$\begin{aligned} (f' * g')(cv_0) &= (f * g)(c) && \text{for } c \in \Gamma \\ (f' * g')(w) &= 0 && \text{unless } w \in \Gamma v_0. \end{aligned}$$

Moreover,  $g'$  is supported on elements of  $\Gamma'$  of shape  $(m, n)$ ,  $\|g'\|_2 = \|g\|_2$ , and  $\|f'\|_2 = \|f\|_2$ . Now apply Theorem 1.9 to the groupoid  $\Gamma'$  and the functions  $f'$  and  $g'$  to complete the proof of Theorem 1.1.  $\square$

Recall from [J] that a group  $G$  has *property (RD)* if there is a length function  $L$  on  $G$  such that any function on  $G$  which is rapidly decreasing relative to  $L$  belongs to the reduced  $C^*$ -algebra of  $G$ .

**COROLLARY 1.10.** *Any group  $\Gamma \leq \text{Aut}(\Delta)$  which acts freely on  $\mathcal{V}_\Delta$  has property (RD).*

*Proof.* The index of  $\text{Aut}_S(\Delta)$  in  $\text{Aut}(\Delta)$  is at most 2. Hence the index of  $\Gamma \cap \text{Aut}_S(\Delta)$  in  $\Gamma$  is at most 2. As property (RD) holds for a group whenever it holds for some subgroup of finite index [J, Proposition 2.1.5], we may assume that  $\Gamma \leq \text{Aut}_S(\Delta)$ .

Fix  $v_0 \in \mathcal{V}_\Delta$  and define  $\sigma(c) = \sigma(v_0, cv_0)$  for  $c \in \Gamma$ . Likewise, define

$$|c| = d(v_0, cv_0).$$

Then  $|c|$  is a length function on  $\Gamma$ , in the sense of [J]. When  $\sigma(c) = (m, n)$  one has  $|c| = m + n$ . To derive property (RD) from Theorem 1.1 we note that a given value of  $|c|$  corresponds to only  $|c| + 1$  different shapes, and that the relevant function  $p(m, n)$  from (1) can be bounded by a polynomial in  $m + n$ . Now argue as in [Co, Chapter III.5, Theorem 5] or [H, Lemma 1.5]. □

**REMARK 1.11.** Corollary 1.10 implies property (RD) for any discrete subgroup of  $\text{Aut}(\Delta)$  which has a torsion-free subgroup of finite index. As a consequence, if  $\mathbb{F}$  is a finite extension of  $\mathbb{Q}_p$  and  $\Gamma$  is a finitely generated discrete subgroup of  $\text{SL}_3(\mathbb{F})$ , then it follows from Selberg’s Lemma [S] that  $\Gamma$  has a torsion free finite index subgroup, and so satisfies property (RD).

**1.5 The retraction centred at a boundary point of the building.**

The proof of Theorem 1.9 relies heavily on the notion of the retraction of a building centred at a boundary point. We outline a construction based on [B, page 171]. There is also a concise description in [Ron, §9.3]. An infinite straight line of edges in an apartment is called a *wall*. Fix an apartment  $\mathcal{A}$  of  $\Delta$ . Consider two half-planes in  $\mathcal{A}$  bounded by intersecting walls and measure the angle between those two walls through the intersection of the two half-planes. If that angle is nonzero and minimal, we call the intersection of the two half-planes a *sector*. A *sector* in  $\Delta$  is by definition a sector in some apartment of  $\Delta$ . The *boundary*,  $\Omega$ , of  $\Delta$  is the set of sectors in  $\Delta$  under the equivalence relation that  $\mathcal{S} \sim \mathcal{S}'$  if  $\mathcal{S} \cap \mathcal{S}'$  contains a sector. (This is the analogue in higher rank of the notion of tail equivalence for semi-infinite paths in trees.) Given a vertex  $v \in \mathcal{V}_\Delta$  and a boundary point  $\omega \in \Omega$  there is a unique sector based at  $v$  representing  $\omega$ , denoted  $\mathcal{S}_v(\omega)$ . Thus  $\Omega$  can be identified with the set of sectors emanating from any fixed vertex in  $\Delta$ .

Fix an apartment  $\mathcal{A}_0$  of  $\Delta$  and a sector  $\mathcal{S}_0$  in  $\mathcal{A}_0$  representing a boundary point  $\omega_0 \in \Omega$ . Then  $\Delta$  is a union of apartments  $\mathcal{A}'$  each of which contains

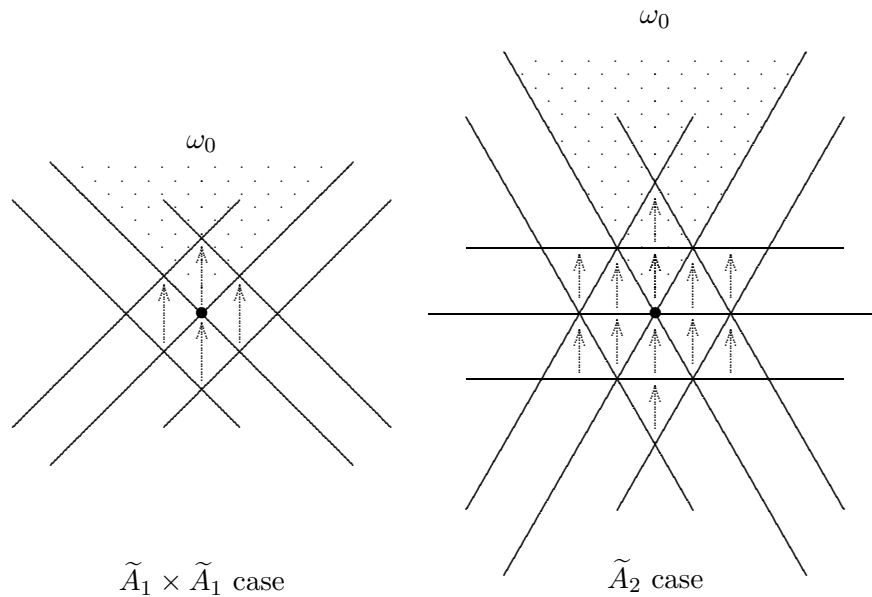


Figure 2: Arrows indicating preferred direction.

a subsector of  $\mathcal{S}_0$ . There is a retraction  $r : \Delta \rightarrow \mathcal{A}_0$  which is a contraction, in the sense that  $d(r(u), r(v)) \leq d(u, v)$  for any  $u, v \in \mathcal{V}_\Delta$ , and whose restriction to any such  $\mathcal{A}'$  is an isomorphism from  $\mathcal{A}'$  onto  $\mathcal{A}_0$  which fixes  $\mathcal{A}' \cap \mathcal{A}_0$  pointwise. In particular, given any  $v \in \mathcal{V}_\Delta$ ,  $r(v)$  is independent of the apartment  $\mathcal{A}'$  chosen such that  $v \in \mathcal{A}'$ .

The boundary point  $\omega_0$  defines a preferred direction in  $\mathcal{A}_0$  which we shall call *up*. Each chamber in  $\mathcal{A}_0$  may therefore be labelled with an arrow pointing up, as in Figure 2. This labelling of the chambers of  $\mathcal{A}_0$  induces a labelling of all chambers of  $\Delta$  via the retraction  $r$ . We say a line segment is *horizontal* if it is perpendicular to the up direction. In particular, a horizontal line segment cannot be part of a sector wall for any sector representing  $\omega_0$ .

It is helpful to imagine that  $\Delta$  has been folded flat over  $\mathcal{A}_0$  according to the retraction  $r$  so as to hang down from  $\omega_0$ . Note that, as in Figure 2, each edge of the apartment  $\mathcal{A}_0$  is incident with one chamber which lies above it and one chamber which lies below it. Given any edge  $(u, v)$  of  $\Delta$ , there is a unique chamber of  $\Delta$  containing  $(u, v)$  which retracts to the chamber

lying above  $(r(u), r(v))$  and all other chambers containing  $(u, v)$  retract to the chamber of  $\mathcal{A}_0$  lying below  $(r(u), r(v))$ .

Suppose that  $C$  and  $C'$  are adjacent chambers of  $\Delta$ . Then either  $r(C)$  is adjacent to but distinct from  $r(C')$ , or  $r(C)$  equals  $r(C')$ . In the first case the arrows on the two chambers are parallel and in the second case they mirror each other through the common edge and point in converging directions. If  $\mathcal{A}$  is an apartment of  $\Delta$ , then the retraction  $r : \mathcal{A} \rightarrow \mathcal{A}_0$  need not be injective. The above constraints then lead to overall labellings of its chambers as in Figure 3, with various degenerate cases also possible. This

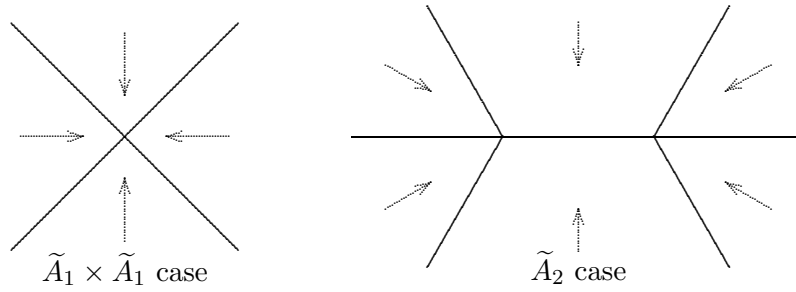


Figure 3: Generic labelling of apartments.

pattern of arrows is called the *folding diagram* for  $\mathcal{A}$ , as it indicates how  $\mathcal{A}$  is folded by  $r$ . The folding diagram is a convenient visual convention for encoding geometric properties of the retraction. In particular, two apartments have the same folding diagram if their retractions are equivalent up to translation in  $\mathcal{A}_0$ . One may similarly discuss the folding diagram of any connected subset of an apartment  $\mathcal{A}$ . Note that in certain degenerate cases it becomes necessary for us to consider the folding diagram of a line segment in  $\Delta$  where there are no chambers for reference. The folding obtained in such a case will be a piecewise straight line and we trust that the reader will have little difficulty in appreciating the possibilities.

For clarity of exposition we will now consider the  $\tilde{A}_1 \times \tilde{A}_1$  and  $\tilde{A}_2$  cases separately. Although the ideas used to tackle the  $\tilde{A}_1 \times \tilde{A}_1$  case are also used in the proof of the  $\tilde{A}_2$  case, the extra complexity of the  $\tilde{A}_2$  case would make a joint exposition unwieldy.

## 2 The $\tilde{A}_1 \times \tilde{A}_1$ Case

*Proof of Theorem 1.9 in the  $\tilde{A}_1 \times \tilde{A}_1$  case.* Let  $W_{m,n} = \{\gamma \in \mathcal{G} \mid \sigma(\gamma) = (m, n)\}$  and fix an element  $g \in \mathbb{C}\mathcal{G}$  whose support is contained in  $W_{m,n}$ , i.e.  $g \in \mathbb{C}W_{m,n}$ . The matrix coefficients of  $\rho(g)$  are given by  $\langle \rho(g)\delta_x, \delta_y \rangle = g(x^{-1}y)$  for  $x, y \in \mathcal{V}_\Delta$  where  $x^{-1}y$  is defined as in §1.3. Thus,

$$\langle \rho(g)\delta_x, \delta_y \rangle = \begin{cases} g(x^{-1}y) & \text{if } x^{-1}y \in W_{m,n} \\ 0 & \text{otherwise.} \end{cases}$$

Geometrically,  $\langle \rho(g)\delta_x, \delta_y \rangle$  is non-zero only if  $x$  and  $y$  are opposite vertices of a rectangle  $P_{m,n}(x, y)$  of the sort pictured in Figure 4. This rectangle

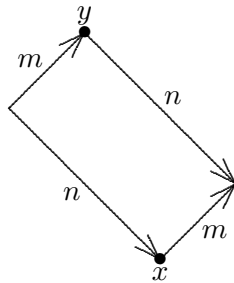


Figure 4: Convex hull of two vertices in  $\Delta$ .

is the convex hull of  $x$  and  $y$ , and as such is common to all apartments containing both  $x$  and  $y$ .

Consider the folding diagram of  $P_{m,n}(x, y)$  induced by the retraction  $r$ . In general this will be as indicated in Figure 5, although certain degeneracies are possible. Let  $z$  be the apex of the upward labelled subrectangle

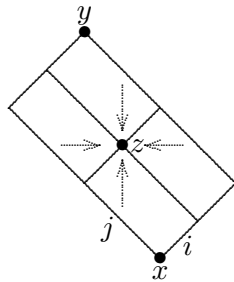


Figure 5: General labelling of a rectangle in  $\Delta$ .

based at  $x$  such that  $x^{-1}z \in W_{i,j}$  with  $i$  and  $j$  maximal, as in the diagram. We call  $z$  the *focal point* of the folding.

Associated with each rectangle  $P_{m,n}(x, y)$  is an abstract diagram  $D(x, y) = D_{i,j}$ , which is a copy of  $P_{m,n}(x, y)$  in which the labels of the vertices are forgotten, but the arrows are retained. Denote by  $\mathcal{D}(m, n)$  the set of all possible diagrams  $D_{i,j}$  for fixed  $m, n \in \mathbb{N}$ .

LEMMA 2.1.       $\#\mathcal{D}(m, n) = (m + 1)(n + 1)$ .

*Proof.* In order to determine the number of possible diagrams it is sufficient to enumerate the possible locations of the focal point of the folding. Hence, there are  $(m + 1)(n + 1)$  possible diagrams. □

For each  $D \in \mathcal{D}(m, n)$ , define an operator  $T_D$  on  $\ell^2(\mathcal{V}_\Delta)$  by

$$\langle T_D \delta_x, \delta_y \rangle = \begin{cases} g(x^{-1}y) & \text{if } D(x, y) = D \\ 0 & \text{otherwise.} \end{cases}$$

Then  $\rho(g) = \sum_{D \in \mathcal{D}(m, n)} T_D$  and so

$$\|\rho(g)\| \leq \sum_{D \in \mathcal{D}(m, n)} \|T_D\|. \tag{2}$$

We now fix  $D = D_{i,j} \in \mathcal{D}(m, n)$  and proceed to estimate  $\|T_D\|$ . For each  $z \in \mathcal{V}_\Delta$  define

$$\mathcal{H}_z = \langle \delta_x \mid x^{-1}z \in W_{i,j} \text{ and } \mathcal{S}_z(\omega_0) \subseteq \mathcal{S}_x(\omega_0) \rangle$$

and

$$\mathcal{K}_z = \langle \delta_y \mid z^{-1}y \in W_{m-i, n-j} \text{ and } \mathcal{S}_z(\omega_0) \subseteq \mathcal{S}_y(\omega_0) \rangle.$$

These give rise to two decompositions;  $\ell^2(\mathcal{V}_\Delta) = \oplus_{z \in \mathcal{V}_\Delta} \mathcal{H}_z = \oplus_{z \in \mathcal{V}_\Delta} \mathcal{K}_z$ . Define an operator  $T_{z,z} : \mathcal{H}_z \rightarrow \mathcal{K}_z$  by

$$\langle T_{z,z} \delta_x, \delta_y \rangle = g(x^{-1}y) \text{ for } x \in \mathcal{H}_z \text{ and } y \in \mathcal{K}_z,$$

and define  $T_{z',z} : \mathcal{H}_z \rightarrow \mathcal{K}_{z'}$  to be zero for  $z' \neq z$ . Then  $T_D$  can be expressed as a block diagonal operator matrix  $T_D = [T_{z',z}] = \oplus_{z \in \mathcal{V}_\Delta} T_{z,z}$  and it is sufficient to show that for fixed  $z \in \mathcal{V}_\Delta$ ,  $\|T_{z,z}\| \leq \|g\|_2$ .

So fix  $z$  and suppose that  $\delta_x \in \mathcal{H}_z$  and  $\delta_y \in \mathcal{K}_z$ . According to Lemma 1.8,  $x$  and  $y$  are uniquely determined by  $x^{-1}y$ . So each  $\gamma \in W_{m,n}$  contributes at most one matrix coefficient of  $T_{z,z}$  in the form  $g(\gamma)$ . Therefore we have  $\|T_{z,z}\| \leq \|T_{z,z}\|_{HS} \leq \|g\|_2$ .

This concludes our proof of Theorem 1.9 in the  $\tilde{A}_1 \times \tilde{A}_1$  case. □

### 3 The $\tilde{A}_2$ Case

*Proof of Theorem 1.9 in the  $\tilde{A}_2$  case.* Let  $W_{m,n} = \{\gamma \in \mathcal{G} : \sigma(\gamma) = (m, n)\}$  and fix an element  $g \in \mathbb{C}\mathcal{G}$  whose support is contained in  $W_{m,n}$ , i.e.  $g \in \mathbb{C}W_{m,n}$ . The matrix coefficients of  $\rho(g)$  are given by  $\langle \rho(g)\delta_x, \delta_y \rangle = g(x^{-1}y)$  for  $x, y \in \mathcal{V}_\Delta$ . Thus,

$$\langle \rho(g)\delta_x, \delta_y \rangle = \begin{cases} g(x^{-1}y) & \text{if } x^{-1}y \in W_{m,n} \\ 0 & \text{otherwise.} \end{cases}$$

Geometrically,  $\langle \rho(g)\delta_x, \delta_y \rangle$  is non-zero only if  $x$  and  $y$  are opposite vertices of a parallelogram  $P_{m,n}(x, y)$  of the sort pictured in Figure 6. This par-

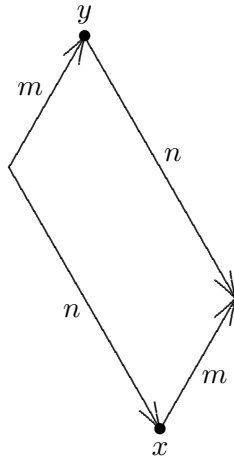


Figure 6: Convex hull of two vertices in  $\Delta$ .

allelogram is the convex hull of  $x$  and  $y$ , and as such is common to all apartments containing both  $x$  and  $y$ . Note that  $x$  and  $y$  are at the acute angles of the parallelogram.

Consider the folding of  $P_{m,n}(x, y)$  induced by the retraction  $r$ . This will be one of the possibilities depicted in Figure 7, with possible degeneracies. Let  $z_x$  be the apex of the upward labelled region based at  $x$  such that  $x^{-1}z_x \in W_{i,j}$  with  $i$  and  $i + j$  maximal. Similarly, let  $z_y$  be the apex of the upward labelled region based at  $y$  such that  $y^{-1}z_y \in W_{k,l}$  with  $k$  and  $k + l$  maximal. In terms of our diagrams,  $z_x$  and  $z_y$  would be as labelled in Figure 8.

We shall refer to  $z_x$  and  $z_y$  as the *focal points* of the folding. If  $z_x \neq z_y$ ,

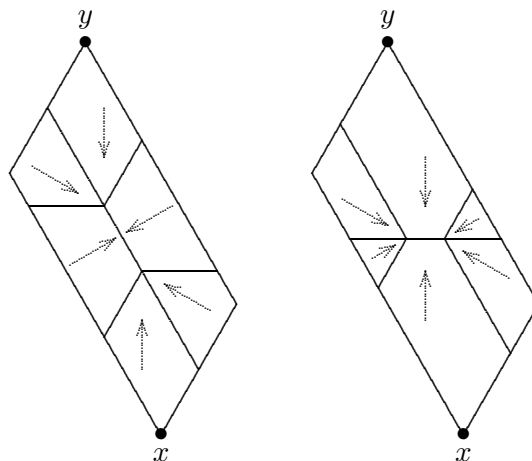


Figure 7: Possible labellings of parallelograms.

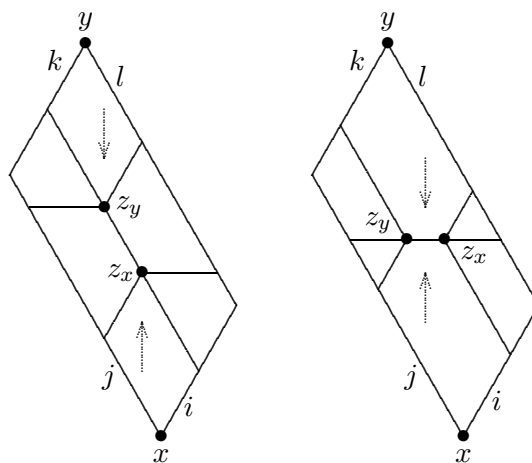


Figure 8: Positions of the vertices  $z_x$  and  $z_y$ .



the convex hull of  $z_x$  and  $z_y$  is a line segment which we denote  $z_x \leftrightarrow z_y$  and the retraction  $r$  maps  $z_x \leftrightarrow z_y$  injectively to a horizontal line segment  $r(z_x \leftrightarrow z_y)$  in  $\mathcal{A}_0$ . Note that the positions of the focal points completely determine the folding. Among possible degeneracies, we could have  $z_x = x$  and  $z_y = y$ , or  $z_x = z_y$ .

Associated to each parallelogram  $P_{m,n}(x,y)$  is an abstract diagram  $D(x,y) = D_{i,j}^{k,l}$ , which is a copy of  $P_{m,n}(x,y)$  in which the labels of the vertices are forgotten, but the arrows are retained as in Figure 9.

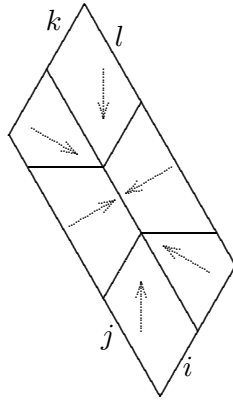


Figure 9: The diagram  $D_{i,j}^{k,l}$ .

Denote by  $\mathcal{D}(m,n)$  the set of all possible diagrams  $D_{i,j}^{k,l}$  for fixed  $m,n \in \mathbb{N}$ . We divide the family  $\mathcal{D}(m,n)$  into two classes as follows. Let  $\mathcal{D}'(m,n)$  be the set of diagrams associated to parallelograms  $P_{m,n}(x,y)$  in which  $x^{-1}z_x \in W_{p,q}$  and  $x^{-1}z_y \in W_{s,t}$  where  $p+q = s+t$ . Thus a diagram  $D \in \mathcal{D}(m,n)$  is in  $\mathcal{D}'(m,n)$  if it is of the form depicted in Figure 10 or if the focal points are coincident. Let  $\mathcal{D}''(m,n) = \mathcal{D}(m,n) \setminus \mathcal{D}'(m,n)$ . So  $D \in \mathcal{D}(m,n)$  is in  $\mathcal{D}''(m,n)$  if it is of one of the forms depicted in Figure 11 with distinct focal points.

LEMMA 3.1.  $\#\mathcal{D}''(m,n) = 1/2(m+1)(n+1)(m+n)$ .

*Proof.* In order to determine the number of diagrams in  $\mathcal{D}''(m,n)$  it is sufficient to enumerate the possible positions of the focal points in the foldings. There are  $(m+1)(n+1)$  choices for the first focal point. Given one focal point, the possible positions of the other are indicated in Figure 12. So there are  $m+n$  positions the second focal point could occupy. Since the

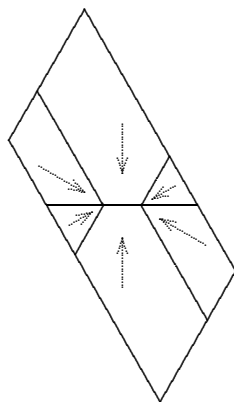


Figure 10: Parallelograms in  $\mathcal{D}'(m, n)$ .

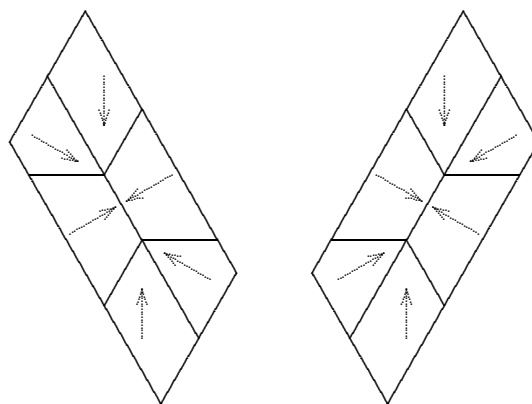


Figure 11: Parallelograms in  $\mathcal{D}''(m, n)$ .

order of choice is irrelevant we must divide the total number of possibilities by a factor of 2. Hence the result.  $\square$

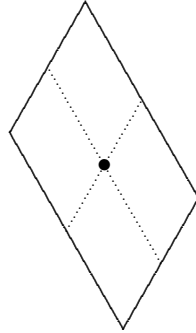


Figure 12: Possible positions of second focal point.

For each  $D \in \mathcal{D}(m, n)$ , define an operator  $T_D$  on  $\ell^2(\mathcal{V}_\Delta)$  by

$$\langle T_D \delta_x, \delta_y \rangle = \begin{cases} g(x^{-1}y) & \text{if } D(x, y) = D \\ 0 & \text{otherwise.} \end{cases}$$

Thus  $\rho(g) = \sum_{D \in \mathcal{D}(m, n)} T_D = \sum_{D \in \mathcal{D}'(m, n)} T_D + \sum_{D \in \mathcal{D}''(m, n)} T_D$ .

For each diagram  $D \in \mathcal{D}'(m, n)$  choose one of the focal points and call it a *distinguished focal point*. There are  $(m + 1)(n + 1)$  possible positions for the distinguished focal point. Let  $\mathcal{D}'_{i,j}(m, n)$  be the set of diagrams in  $\mathcal{D}'(m, n)$  whose distinguished focal point is in position  $(i, j)$ , so that

$$\sum_{D \in \mathcal{D}'(m, n)} T_D = \sum_{i=0}^m \sum_{j=0}^n \sum_{D \in \mathcal{D}'_{i,j}(m, n)} T_D.$$

An argument analogous to that used in §2 shows that

$$\left\| \sum_{D \in \mathcal{D}'_{i,j}(m, n)} T_D \right\| \leq \|g\|_2.$$

Hence

$$\left\| \sum_{D \in \mathcal{D}'(m, n)} T_D \right\| \leq (m + 1)(n + 1)\|g\|_2$$

and so

$$\|\rho(g)\| \leq (m + 1)(n + 1)\|g\|_2 + \sum_{D \in \mathcal{D}''(m, n)} \|T_D\|. \tag{3}$$

We proceed to show that, for  $D \in \mathcal{D}''(m, n)$ ,  $\|T_D\| \leq \sqrt{\max(m, n) + 1} \|g\|_2$ . Since

$$(m + 1)(n + 1) + \frac{1}{2}(m + 1)(n + 1)(m + n) = \frac{1}{2}(m + 1)(n + 1)(m + n + 2)$$

we will then have established that

$$\|\rho(g)\| \leq 1/2(m + 1)(n + 1)(m + n + 2)\sqrt{\max(m, n) + 1} \|g\|_2.$$

We now fix  $D = D_{i,j}^{k,l} \in \mathcal{D}''(m, n)$  and estimate  $\|T_D\|$ . Note that, if  $D(x, y) = D$ , then  $z_x^{-1}z_y \in W_{s,t}$  where  $(s, t) = (p, 0)$  or  $(0, q)$ , and  $(s, t)$  is determined by  $D$ . Thus  $i, j, k, l, s, t$  are all now fixed.

Given  $z \in \mathcal{V}_\Delta$ , define

$$\mathcal{H}_z = \langle \delta_x \mid x^{-1}z \in W_{i,j} \quad \text{and} \quad \mathcal{S}_z(\omega_0) \subseteq \mathcal{S}_x(\omega_0) \rangle$$

and

$$\mathcal{K}_z = \langle \delta_y \mid y^{-1}z \in W_{k,l} \quad \text{and} \quad \mathcal{S}_z(\omega_0) \subseteq \mathcal{S}_y(\omega_0) \rangle.$$

These give rise to two decompositions;  $\ell^2(\mathcal{V}_\Delta) = \bigoplus_{z \in \mathcal{V}_\Delta} \mathcal{H}_z = \bigoplus_{z \in \mathcal{V}_\Delta} \mathcal{K}_z$ .

If  $(z_1, z_2) \in \mathcal{V}_\Delta \times \mathcal{V}_\Delta$  with  $z_1^{-1}z_2 \in W_{s,t}$  and  $r$  maps  $z_1 \leftrightarrow z_2$  injectively to a horizontal line, define  $T_{z_2, z_1} : \mathcal{H}_{z_1} \rightarrow \mathcal{K}_{z_2}$  by

$$\langle T_{z_2, z_1} \delta_x, \delta_y \rangle = g(x^{-1}y) \text{ for } \delta_x \in \mathcal{H}_{z_1} \text{ and } \delta_y \in \mathcal{K}_{z_2},$$

and define  $T_{z_2, z_1} : \mathcal{H}_{z_1} \rightarrow \mathcal{K}_{z_2}$  to be zero for other pairs  $(z_1, z_2) \in \mathcal{V}_\Delta \times \mathcal{V}_\Delta$ .

Then  $T_D$  can be expressed as an operator matrix  $T_D = [T_{z_2, z_1}]$  and therefore

$$\|T_D\| \leq \left\| \left[ \|T_{z_2, z_1}\| \right] \right\| \tag{4}$$

by Lemma 1.2. We proceed to estimate  $\|T_{z_2, z_1}\|$ .

Fix  $(z_1, z_2) \in \mathcal{V}_\Delta \times \mathcal{V}_\Delta$  with  $z_1^{-1}z_2 \in W_{s,t}$  and such that  $r$  maps  $z_1 \leftrightarrow z_2$  injectively to a horizontal line. Thus  $r(z_1 \leftrightarrow z_2)$  is a horizontal line of shape  $(s, t)$ . The vectors  $\delta_x \in \mathcal{H}_{z_1}$ ,  $\delta_y \in \mathcal{K}_{z_2}$  are uniquely determined by  $x^{-1}y \in \mathcal{G}$  according to Lemma 1.8. Thus each  $\gamma \in W_{m,n}$  contributes at most one matrix coefficient of  $T_{z_2, z_1}$  in the form  $g(\gamma)$ . Define  $\tilde{g} \in \mathbb{C}W_{s,t}$  by

$$\tilde{g}(\zeta) = \begin{cases} \left( \sum_{\substack{\alpha \in W_{i,j}, \beta \in W_{k,l} \\ \alpha\zeta\beta \in W_{m,n}}} |g(\alpha\zeta\beta)|^2 \right)^{1/2} & \text{if } \zeta \in W_{s,t} \\ 0 & \text{otherwise.} \end{cases}$$

Note that  $\|\tilde{g}\|_2 = \|g\|_2$  and

$$\|T_{z_2, z_1}\| \leq \|T_{z_2, z_1}\|_{HS} \leq \tilde{g}(z_1^{-1}z_2). \tag{5}$$

Define a new operator  $\tilde{T}_D$  on  $\ell^2(\mathcal{V}_\Delta)$  by

$$\langle \tilde{T}_D \delta_{z_1}, \delta_{z_2} \rangle = \begin{cases} \tilde{g}(z_1^{-1}z_2) & \text{if } z_1^{-1}z_2 \in W_{s,t} \text{ and } r(z_1 \leftrightarrow z_2) \text{ is} \\ & \text{a horizontal line of shape } (s, t), \\ 0 & \text{otherwise.} \end{cases}$$

By (4), (5) and Lemma 1.3 we have  $\|T_D\| \leq \|\tilde{T}_D\|$ . Since we also have  $\|\tilde{g}\|_2 = \|g\|_2$ , it will be sufficient to prove

$$\|\tilde{T}_D\| \leq \sqrt{\max(m, n) + 1} \|\tilde{g}\|_2. \tag{6}$$

Recall that  $(s, t) = (p, 0)$  with  $0 \leq p \leq m$  or  $(0, q)$  with  $0 \leq p \leq n$ . We suppose the former and prove that

$$\|\tilde{T}_D\| \leq \sqrt{p + 1} \|\tilde{g}\|_2. \tag{7}$$

A similar argument in the case  $(s, t) = (0, q)$  gives  $\|\tilde{T}_D\| \leq \sqrt{q + 1} \|\tilde{g}\|_2$  thus establishing (6).

In order to prove (7), we simplify our notation. Let  $g \in \mathbb{C}W_{p,0}$  where  $0 \leq p \leq m$  and define an operator  $T$  on  $\ell^2(\mathcal{V}_\Delta)$  by

$$\langle T\delta_x, \delta_y \rangle = \begin{cases} g(x^{-1}y) & \text{if } x^{-1}y \in W_{p,0} \text{ and } r(x \leftrightarrow y) \text{ is} \\ & \text{a horizontal line of shape } (p, 0), \\ 0 & \text{otherwise.} \end{cases}$$

We must prove

$$\|T\| \leq \sqrt{p + 1} \|g\|_2. \tag{8}$$

Given  $(x, y) \in \mathcal{V}_\Delta \times \mathcal{V}_\Delta$  with  $x^{-1}y \in W_{p,0}$  and such that  $r(x \leftrightarrow y)$  is a horizontal line of shape  $(p, 0)$ , there is a unique  $z = z(x, y) \in \mathcal{V}_\Delta$  such that the convex hull of  $x, y$  and  $z$  is labelled as in Figure 13 and such that  $z^{-1}x, x^{-1}y, y^{-1}z \in W_{p,0}$ . That is  $xyz$  is an equilateral triangle, with side of length  $p$ , base  $xy$  and pointing to  $\omega_0$ . To see this, suppose that  $x_1$  is the

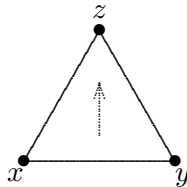


Figure 13: Labelling of convex hull of  $x, y$  and  $z = z(x, y)$ .

vertex adjacent to, but distinct from,  $x$  in  $x \leftrightarrow y$ . Let  $w$  be the third vertex of the unique chamber containing  $(x, x_1)$  and retracting above  $(r(x), r(x_1))$ . The convex hull of  $x, y$  and  $w$  is a trapezoidal strip in  $\Delta$  which has  $x \leftrightarrow y$  as one of its bases, contains the chamber with vertices  $(x, x_1, w)$  and whose other base,  $w \leftrightarrow w'$ , has length  $p - 1$  as in Figure 14. Since the retraction is contractive,  $r$  must map  $w \leftrightarrow w'$  injectively to a horizontal line lying

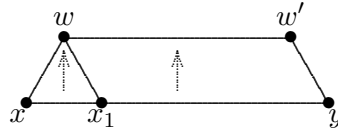


Figure 14: Trapezoidal strip retracting above  $x \leftrightarrow y$ .

above  $r(x) \leftrightarrow r(y)$ . By induction, this argument demonstrates the existence of the triangle depicted in Figure 13. Note that  $\mathcal{S}_z(\omega_0) = \mathcal{S}_x(\omega_0) \cap \mathcal{S}_y(\omega_0)$ . For later convenience we define a set

$$\mathcal{C} = \left\{ (x, y, z) \in \mathcal{V}_\Delta \times \mathcal{V}_\Delta \times \mathcal{V}_\Delta \mid x^{-1}y \in W_{p,0}, r(x \leftrightarrow y) \text{ is a horizontal line of shape } (p, 0), \text{ and } \mathcal{S}_z(\omega_0) = \mathcal{S}_x(\omega_0) \cap \mathcal{S}_y(\omega_0) \right\}.$$

For each  $z \in \mathcal{V}_\Delta$ , define

$$\mathcal{H}_z = \langle \delta_x \mid z^{-1}x \in W_{p,0} \text{ and } \mathcal{S}_z(\omega_0) \subseteq \mathcal{S}_x(\omega_0) \rangle$$

and

$$\mathcal{K}_z = \langle \delta_y \mid y^{-1}z \in W_{p,0} \text{ and } \mathcal{S}_z(\omega_0) \subseteq \mathcal{S}_y(\omega_0) \rangle.$$

Thus  $\delta_x \in \mathcal{H}_z$  if  $z = z(x, y)$  for some  $y \in \mathcal{V}_\Delta$  and a similar condition characterizes the elements  $\delta_y \in \mathcal{K}_z$ . Let  $T_{z',z} : \mathcal{H}_z \rightarrow \mathcal{K}_{z'}$  be defined by

$$\langle T_{z',z} \delta_x, \delta_y \rangle = g(x^{-1}y) \text{ if } \delta_x \in \mathcal{H}_z \text{ and } \delta_y \in \mathcal{K}_{z'}.$$

Note that  $\ell^2(\mathcal{V}_\Delta) = \bigoplus_{z \in \mathcal{V}_\Delta} \mathcal{H}_z = \bigoplus_{z' \in \mathcal{V}_\Delta} \mathcal{K}_{z'}$ . Since  $T_{z',z} = 0$  unless  $z' = z$ ,  $T$  can be expressed as a block diagonal matrix  $T = [T_{z',z}] = \bigoplus_{z \in \mathcal{V}_\Delta} T_{z,z}$ . Thus it is sufficient to show that, for fixed  $z \in \mathcal{V}_\Delta$ ,

$$\|T_{z,z}\| \leq \sqrt{p+1} \|g\|_2. \tag{9}$$

We now fix  $z$  and show that

$$|\langle T_{z,z} f_1, f_2 \rangle| \leq \sqrt{p+1} \|f_1\|_2 \|f_2\|_2 \|g\|_2$$

for  $f_1 \in \mathcal{H}_z$ , and  $f_2 \in \mathcal{K}_z$ . We may assume  $f_1, f_2, g \geq 0$ . Then

$$\langle T_{z,z} f_1, f_2 \rangle = \sum_{\substack{x, y \in \mathcal{V}_\Delta \\ (x, y, z) \in \mathcal{C}}} f_1(x) f_2(y) g(x^{-1}y).$$

We define  $g_1, g_2 \in \mathbb{C}\mathcal{G}$  via

$$g_1(\alpha) = \begin{cases} f_1(x) & \text{if } \alpha = z^{-1}x \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad g_2(\beta) = \begin{cases} f_2(y) & \text{if } \beta = y^{-1}z \\ 0 & \text{otherwise} \end{cases}$$

and change our emphasis to obtain

$$\langle T_{z,z}f_1, f_2 \rangle = \sum_{\substack{x,y \in \mathcal{V}_\Delta \\ (x,y,z) \in \mathcal{C}}} g_1(z^{-1}x)g_2(y^{-1}z)g(x^{-1}y). \tag{10}$$

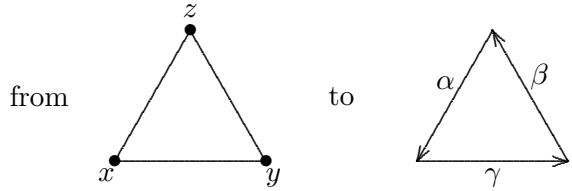
We define a set

$$\mathcal{T}_p = \{(\alpha, \beta, \gamma) \in \mathcal{G} \times \mathcal{G} \times \mathcal{G} \mid \alpha, \beta, \gamma \in W_{p,0} \text{ and } \gamma\beta\alpha \text{ is a unit}\}$$

It can be shown that if  $x, y, z$  are vertices in an  $\tilde{A}_2$  building and  $\sigma(x, y) = \sigma(y, z) = \sigma(z, x) = (p, 0)$ , the convex hull of  $\{x, y, z\}$  is necessarily an equilateral triangle of side length  $p$  in some apartment. Hence, given any  $(\alpha, \beta, \gamma) \in \mathcal{T}_p$  and any  $v \in \mathcal{V}_\Delta$  such that  $v\gamma$  exists, the vertices  $v, v\gamma, v\alpha^{-1}$  lie in a common apartment and their convex hull is an equilateral triangle of side length  $p$ . Use  $\mathcal{T}_p$  to rewrite equation (10) as

$$\langle T_{z,z}f_1, f_2 \rangle = \sum_{(\alpha, \beta, \gamma) \in \mathcal{T}_p} \tilde{f}_1(\alpha)\tilde{f}_2(\beta)\tilde{f}_3(\gamma),$$

where  $\tilde{f}_i \in \mathbb{C}W_{p,0}$  for  $i = 1, 2, 3$ . This is equivalent to having changed the emphasis



We now complete the proof of (9) by proving the following result.

LEMMA 3.2. *If  $f_1, f_2, f_3 \in \mathbb{C}W_{p,0}$  then*

$$\left| \sum_{(\alpha, \beta, \gamma) \in \mathcal{T}_p} f_1(\alpha)f_2(\beta)f_3(\gamma) \right| \leq \sqrt{p+1} \|f_1\|_2 \|f_2\|_2 \|f_3\|_2.$$

*Proof.* Assume that  $f_1, f_2, f_3 \geq 0$ . For a fixed  $f_3$ , define  $T_3 : \mathbb{C}W_{p,0} \rightarrow \mathbb{C}W_{p,0}$  by

$$\langle T_3\delta_\alpha, \delta_\beta \rangle = \sum_{\gamma: (\alpha, \beta, \gamma) \in \mathcal{T}_p} f_3(\gamma)$$

where the right-hand side has at most one non-zero term, and we adopt the convention that it is zero if the set  $\{\gamma \in W_{p,0} \mid (\alpha, \beta, \gamma) \in \mathcal{T}_p\}$  is empty. We will use this convention again later. Thus

$$\sum_{(\alpha, \beta, \gamma) \in \mathcal{T}_p} f_1(\alpha)f_2(\beta)f_3(\gamma) = \langle T_3f_1, f_2 \rangle.$$

We must show that  $\|T_3\| \leq \sqrt{p+1} \|f_3\|_2$ , or equivalently that  $\|T_3^*T_3\| \leq (p+1) \|f_3\|_2^2$ . Now,

$$\begin{aligned} \langle T_3^*T_3\delta_{\alpha_1}, \delta_{\alpha_2} \rangle &= \langle T_3\delta_{\alpha_1}, T_3\delta_{\alpha_2} \rangle \\ &= \sum_{\beta \in \mathcal{G}} \left( \sum_{\gamma_1: (\alpha_1, \beta, \gamma_1) \in \mathcal{T}_p} f_3(\gamma_1) \right) \left( \sum_{\gamma_2: (\alpha_2, \beta, \gamma_2) \in \mathcal{T}_p} f_3(\gamma_2) \right) \\ &= \sum_{\substack{\beta, \gamma_1, \gamma_2 \in \mathcal{G} \\ (\alpha_1, \beta, \gamma_1), (\alpha_2, \beta, \gamma_2) \in \mathcal{T}_p}} f_3(\gamma_1)f_3(\gamma_2). \end{aligned}$$

We say that  $\alpha_1, \alpha_2 \in \mathcal{G}$  share precisely  $j$  initial letters if

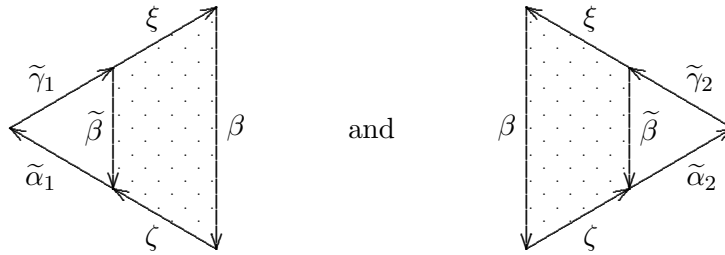
- there exists a  $z \in \mathcal{V}_\Delta$  such that  $z\alpha_1$  and  $z\alpha_2$  are both defined, and
- there exist  $\zeta \in W_{j,0}$ , and  $\tilde{\alpha}_1, \tilde{\alpha}_2 \in W_{p-j,0}$  such that  $\alpha_1 = \zeta\tilde{\alpha}_1$ ,  $\alpha_2 = \zeta\tilde{\alpha}_2$ , and

$$\sigma(\tilde{\alpha}_1^{-1}\tilde{\alpha}_2) = (p-j, p-j).$$

Note that if  $\alpha_1, \alpha_2 \in W_{p,0}$ , and if  $\alpha_1^{-1}\alpha_2$  is defined, then  $\alpha_1$  and  $\alpha_2$  must share precisely  $j$  letters for some  $0 \leq j \leq p$ . We have a decomposition  $T_3^*T_3 = S_0 + \dots + S_p$  where

$$\langle S_j\delta_{\alpha_1}, \delta_{\alpha_2} \rangle = \begin{cases} \langle T_3^*T_3\delta_{\alpha_1}, \delta_{\alpha_2} \rangle & \text{if } \alpha_1, \alpha_2 \text{ share precisely } j \text{ initial letters,} \\ 0 & \text{otherwise.} \end{cases}$$

It is therefore sufficient to prove that  $\|S_j\| \leq \|f_3\|_2^2$  for  $0 \leq j \leq p$ . Diagrammatically speaking,  $\langle S_j\delta_{\alpha_1}, \delta_{\alpha_2} \rangle \neq 0$  only if there are diagrams



whose common (shaded) sections are identical.

For a fixed  $\zeta \in W_{j,0}$ , let

$$\mathcal{H}_\zeta = \langle \delta_\alpha \mid \alpha \in W_{p,0}, \alpha = \zeta\tilde{\alpha}, \tilde{\alpha} \in W_{p-j,0} \rangle.$$

If  $\delta_\alpha \in \mathcal{H}_\zeta$  then  $\langle S_j\delta_\alpha, \delta_{\alpha'} \rangle = 0$  unless  $\delta_{\alpha'} \in \mathcal{H}_\zeta$ . Therefore  $S_j\mathcal{H}_\zeta \subseteq \mathcal{H}_\zeta$  and it follows that  $S_j = \bigoplus_{\zeta \in W_{j,0}} S_j^\zeta$ , where  $S_j^\zeta : \mathcal{H}_\zeta \rightarrow \mathcal{H}_\zeta$  is the restriction of  $S_j$  to  $\mathcal{H}_\zeta$ . So it is enough to bound  $S_j^\zeta$  for each  $\zeta \in W_{j,0}$ . However, in this



case

$$\begin{aligned} \|S_j^\zeta\|^2 &\leq \|S_j^\zeta\|_{HS}^2 = \sum_{\substack{\alpha_1, \alpha_2 \in \Gamma \\ \alpha_1 = \zeta \tilde{\alpha}_1 \\ \alpha_2 = \zeta \tilde{\alpha}_2 \\ \tilde{\alpha}_1^{-1} \tilde{\alpha}_2 \in W_{p-j, p-j}}} \left( \sum_{\substack{\beta, \gamma_1, \gamma_2 \in W_{p,0} \\ (\alpha_1, \beta, \gamma_1) \in \mathcal{T}_p \\ (\alpha_2, \beta, \gamma_2) \in \mathcal{T}_p}} f_3(\gamma_1) f_3(\gamma_2) \right)^2 \\ &= \sum_{\substack{\tilde{\alpha}_1, \tilde{\alpha}_2 \in W_{p-j,0} \\ \tilde{\alpha}_1^{-1} \tilde{\alpha}_2 \in W_{p-j, p-j}}} \left( \sum_{\substack{\xi \in W_{j,0}, \tilde{\beta}, \tilde{\gamma}_1, \tilde{\gamma}_2 \in W_{p-j,0} \\ \tilde{\gamma}_1 \xi, \tilde{\gamma}_2 \xi, \xi^{-1} \tilde{\beta} \zeta^{-1} \in W_{p,0} \\ (\tilde{\alpha}_1, \tilde{\beta}, \tilde{\gamma}_1), (\tilde{\alpha}_2, \tilde{\beta}, \tilde{\gamma}_2) \in \mathcal{T}_{p-j}}} f_3(\tilde{\gamma}_1 \xi) f_3(\tilde{\gamma}_2 \xi) \right)^2, \end{aligned}$$

where the last term is obtained from the previous one by using the smaller triangles formed by  $\tilde{\alpha}_i$ ,  $\tilde{\beta}$ , and  $\tilde{\gamma}_i$  to enumerate the larger ones formed by  $\alpha_i$ ,  $\beta$ , and  $\gamma_i$ . Thus

$$\|S_j^\zeta\|^2 \leq \sum_{\substack{\tilde{\alpha}_1, \tilde{\alpha}_2 \in W_{p-j,0} \\ \tilde{\alpha}_1^{-1} \tilde{\alpha}_2 \in W_{p-j, p-j}}} \left( \sum_{\substack{\tilde{\beta}, \tilde{\gamma}_1, \tilde{\gamma}_2 \in W_{p-j,0} \\ (\tilde{\alpha}_1, \tilde{\beta}, \tilde{\gamma}_1) \in \mathcal{T}_{p-j} \\ (\tilde{\alpha}_2, \tilde{\beta}, \tilde{\gamma}_2) \in \mathcal{T}_{p-j}}} \left( \sum_{\substack{\xi \in W_{j,0} \\ \tilde{\gamma}_1 \xi, \tilde{\gamma}_2 \xi \in W_{p,0}}} f_3(\tilde{\gamma}_1 \xi) f_3(\tilde{\gamma}_2 \xi) \right) \right)^2 \tag{11}$$

where the middle sum is over all possible diagrams of the form depicted in Figure 15. Since  $\tilde{\alpha}_1^{-1} \tilde{\alpha}_2 \in W_{p-j, p-j}$ , an application of Lemma 1.8 shows that this rhombus is entirely determined by  $\tilde{\alpha}_1$  and  $\tilde{\alpha}_2$  and so that the middle sum contains only one term. The rhombus in Figure 15 is also

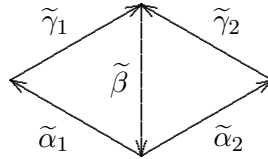


Figure 15: Diagram indexing the middle sum in (11).

uniquely determined by  $\tilde{\gamma}_1$  and  $\tilde{\gamma}_2$  whenever  $\tilde{\gamma}_1 \tilde{\gamma}_2^{-1} \in W_{p-j, p-j}$ , so we in fact have

$$\|S_j^\zeta\|^2 \leq \sum_{\substack{\tilde{\gamma}_1, \tilde{\gamma}_2 \in W_{p-j,0} \\ \tilde{\gamma}_1 \tilde{\gamma}_2^{-1} \in W_{p-j, p-j}}} \left( \sum_{\substack{\xi \in W_{j,0} \\ \tilde{\gamma}_1 \xi, \tilde{\gamma}_2 \xi \in W_{p,0}}} f_3(\tilde{\gamma}_1 \xi) f_3(\tilde{\gamma}_2 \xi) \right)^2.$$

Applying the Cauchy-Schwarz inequality we thus obtain

$$\begin{aligned} \|S_j^\zeta\|^2 &\leq \sum_{\substack{\tilde{\gamma}_1, \tilde{\gamma}_2 \in W_{p-j,0} \\ \tilde{\gamma}_1 \tilde{\gamma}_2^{-1} \in W_{p-j,p-j}}} \left( \sum_{\substack{\xi_1 \in W_{j,0} \\ \xi_1 \tilde{\gamma}_1 \in W_{p,0}}} f_3(\tilde{\gamma}_1 \xi_1)^2 \right) \left( \sum_{\substack{\xi_2 \in W_{j,0} \\ \tilde{\gamma}_2 \xi_2 \in W_{p,0}}} f_3(\tilde{\gamma}_2 \xi_2)^2 \right) \\ &\leq \left( \sum_{\gamma_1 \in \mathcal{G}} f_3(\gamma_1)^2 \right) \left( \sum_{\gamma_2 \in \mathcal{G}} f_3(\gamma_2)^2 \right) \\ &= \|f_3\|_2^4 \end{aligned}$$

as desired. □

This concludes our proof of Theorem 1.9 in the  $\tilde{A}_2$  case. □

### 4 Simply transitive actions

Suppose that  $\Gamma \leq \text{Aut}_S(\Delta)$  acts simply transitively on  $\mathcal{V}_\Delta$ . Fix any vertex  $v_0 \in \mathcal{V}_\Delta$  and let

$$N = \{a \in \Gamma \mid d(v_0, av_0) = 1\}.$$

The *Cayley graph* of  $\Gamma$  constructed via right multiplication with respect to the set  $N$  has  $\Gamma$  itself as its vertex set and has  $\{(c, ca) \mid c \in \Gamma, a \in N\}$  as its edge set. There is a natural action of  $\Gamma$  on its Cayley graph via left multiplication. Using the convention that an undirected edge between vertices  $u$  and  $v$  in a graph represents the pair of directed edges  $(u, v)$  and  $(v, u)$ , it is immediate that the  $\Gamma$ -map  $c \mapsto cv_0$  from  $\Gamma$  to  $\Delta$  is an isomorphism between the Cayley graph of  $\Gamma$  and the one-skeleton of  $\Delta$ . In this way we identify  $\Gamma$  with the vertex set  $\mathcal{V}_\Delta$  of the building. Connectivity of the building implies that  $N$  is a generating set for  $\Gamma$ .

It is traditional to label the directed edge  $(c, ca)$  with the generator  $a \in N$ . More generally, to the pair  $(c, d) \in \Gamma \times \Gamma$  we assign the label  $c^{-1}d$ . Equivalently, to the pair  $(c, cd)$  we assign the label  $d \in \Gamma$ . Suppose this label is written as a product of generators;  $d = a_1 \cdots a_j$ . Then there is a path  $(c, ca_1, ca_1 a_2, \dots, cd)$  from  $c$  to  $cd$  whose successive edges are labelled  $a_1, \dots, a_j$ . The left translate of  $(c, cd)$  by  $b \in \Gamma$  is  $(bc, bcd)$  and also carries the label  $d$ . Conversely, any pair  $(c', c'd)$  which carries the label  $d$  is the left translate by  $b = c'c^{-1}$  of  $(c, cd)$ . Thus two pairs carry the same label if and only if one is the left translate of the other.

We define a shape function on  $\Gamma$  by

$$\sigma(b) = \sigma(v_0, bv_0)$$

for  $b \in \Gamma$ . The pair  $(c, cb)$  has label  $b$  and its shape, defined via the identification of the Cayley graph and the one-skeleton of  $\Delta$ , is

$$\sigma(c, cb) = \sigma(cv_0, cbv_0) = \sigma(v_0, bv_0) = \sigma(b).$$

A different choice of  $v_0$  leads to a shape function on  $\Gamma$  which differs from the first by precomposition with an inner automorphism of  $\Gamma$ .

Suppose  $\Delta$  is an  $\tilde{A}_{n_1} \times \cdots \times \tilde{A}_{n_k}$  building and that  $\Gamma \leq \text{Aut}(\Delta)$  consists of type-rotating automorphisms and acts simply transitively on  $\mathcal{V}_\Delta$ . Then  $\Gamma$  is called an  $\tilde{A}_{n_1} \times \cdots \times \tilde{A}_{n_k}$  group.

**4.1 The case of simply transitive group actions on the vertices of  $\tilde{A}_1 \times \tilde{A}_1$  buildings.** Suppose that  $\Delta = \Delta_1 \times \Delta_2$  is an  $\tilde{A}_1 \times \tilde{A}_1$  building and  $a \in \text{Aut}_S(\Delta)$ . If we write  $(u, v)$  for a generic vertex of  $\Delta = \Delta_1 \times \Delta_2$  we have

$$a(u, v) = (a_1u, a_2v)$$

for some type rotating automorphisms  $a_i$  of  $\Delta_i$ . Indeed, suppose that  $(u, v)$  and  $(u, v')$  are neighbouring vertices in  $\Delta$ . Then  $a(u, v) = (x, y)$  and  $a(u, v') = (x', y')$  are neighbouring vertices in  $\Delta$  and the type-rotating assumption on  $a$  means that  $\tau(x) = \tau(x')$ . Since neighbouring vertices in  $\Delta_1$  have distinct types we must have  $x = x'$ . By induction on  $d(v, v')$ , we see that the first coordinate of  $a(u, v)$  is independent of  $v \in \Delta_2$ . Similarly, the second coordinate of  $a(u, v)$  is independent of  $u \in \Delta_1$ . Thus there exist maps  $a_1$  of  $\Delta_1$  and  $a_2$  of  $\Delta_2$  such that  $a(u, v) = (a_1u, a_2v)$ . Since  $a \in \text{Aut}_S(\Delta)$  it follows that  $a_i$  is a (type-rotating) automorphism of  $\Delta_i$ . Thus each  $a \in \text{Aut}_S(\Delta)$  acts as  $a_1 \times a_2$  for some (type-rotating) automorphisms  $a_i$  of  $\Delta_i$ .

Consider an  $\tilde{A}_1 \times \tilde{A}_1$  group, that is a group  $\Gamma \leq \text{Aut}_S(\Delta)$  which acts simply transitively on  $\mathcal{V}_\Delta$ . A simple example of such a group is  $\mathbb{F}_p \times \mathbb{F}_q$  where  $\mathbb{F}_j$  denotes the free group on  $j$  generators. More generally,  $\Gamma = \Gamma_1 \times \Gamma_2$  is an  $\tilde{A}_1 \times \tilde{A}_1$  group whenever  $\Gamma_1$  and  $\Gamma_2$  are  $\tilde{A}_1$  groups. However there are also many examples of  $\tilde{A}_1 \times \tilde{A}_1$  groups generated by  $\tilde{A}_1$  groups  $\Gamma_1$  and  $\Gamma_2$  where the elements of  $\Gamma_1$  do not all commute with those of  $\Gamma_2$ . In fact S. Mozes [Moz, Theorem 3.2] has given examples of  $\tilde{A}_1$  groups  $\Gamma_1, \Gamma_2$  acting on trees  $T_1, T_2$  which are embedded “diagonally” in  $\text{Aut}(T_1 \times T_2)$ , so that the embeddings do not commute but the group generated by  $\Gamma_1 \cup \Gamma_2$  is an  $\tilde{A}_1 \times \tilde{A}_1$  group acting on  $T_1 \times T_2$ . In other words, even though each  $a \in \Gamma$  is a direct product of automorphisms,  $\Gamma$  is not a direct product of the groups  $\Gamma_1$  and  $\Gamma_2$ . In a further remarkable development, M. Burger and S. Mozes [BuMoz] have shown that the class of  $\tilde{A}_1 \times \tilde{A}_1$  groups contains some partic-

ularly interesting groups: for example finitely presented torsion-free simple groups. The mere existence of such groups had been an outstanding question in group theory.

As above, fix a vertex  $v_0 = (v_1, v_2) \in \mathcal{V}_\Delta$  and suppose that  $b = b_1 \times b_2 \in \Gamma$ . Recall that

$$\sigma(b) = \sigma(v_0, bv_0) = (d(v_1, b_1v_1), d(v_2, b_2v_2)),$$

so that

$$\sigma(b_1 \times b_2) = (\sigma_1(b_1), \sigma_2(b_2))$$

where  $\sigma_i$  is the shape function on  $\text{Aut}(\Delta_i)$ . Consider the generating set

$$N = \{a \in \Gamma \mid d(v_0, av_0) = 1\}$$

of  $\Gamma$ . Let

$$N_1 = \{a \in \Gamma \mid \sigma(a) = (1, 0)\} \quad \text{and} \quad N_2 = \{a \in \Gamma \mid \sigma(a) = (0, 1)\}.$$

Then each element  $c \in \Gamma$  has a unique reduced expression of the form

$$c = a_1 \cdots a_m b_{m+1} \cdots b_{m+n}$$

for some  $a_i \in N_1$  and  $b_i \in N_2$  and the shape function satisfies

$$\sigma(a_1 \cdots a_m b_{m+1} \cdots b_{m+n}) = (m, n).$$

**COROLLARY 4.1.** *If  $\Gamma$  is an  $\tilde{A}_1 \times \tilde{A}_1$  group,  $\rho$  is the right regular representation of  $\ell^1(\Gamma)$  on  $\ell^2(\Gamma)$  defined by  $\rho(g)f = f * g$ , and  $g \in \mathbb{C}\Gamma$  is supported on words of shape  $(m, n)$  then*

$$\|\rho(g)\| \leq (m + 1)(n + 1) \|g\|_2.$$

Consequently  $\Gamma$  has property (RD).

*Proof.* These results follow from Theorem 1.1 and Corollary 1.10. □

**REMARK 4.2.** Since  $\mathbb{Z}^2$  and the free group  $\mathbb{F}_2$  can both be realized as subgroups of  $\tilde{A}_1 \times \tilde{A}_1$  groups, our approach provides a unified method to establish property (RD) for these groups. This answers a question in [Ha2, Section 6].

**4.2 The case of simply transitive group actions on the vertices of  $\tilde{A}_2$  buildings.** Suppose now that  $\Delta$  is an  $\tilde{A}_2$  building and  $\Gamma \leq \text{Aut}_s(\Delta)$  acts simply transitively on  $\mathcal{V}_\Delta$ . Recall that such a group  $\Gamma$  is called an  $\tilde{A}_2$  group. A detailed account of  $\tilde{A}_2$  groups may be found in [CMStZ], while [RRo] and [RoSt] have quick introductions to  $\tilde{A}_2$  buildings and groups. Some, but not all,  $\tilde{A}_2$  groups can be embedded as lattices in matrix groups of the form  $PGL_3(K)$  where  $K$  is a local field [CMStZ, II §8].

For an  $\tilde{A}_2$  group  $\Gamma$ , the shape function  $\sigma$  defined in §4 has an illuminating algebraic interpretation. Recall that for any fixed vertex  $v_0 \in \mathcal{V}_\Delta$  the set

$$N = \{a \in \Gamma \mid \sigma(v_0, av_0) = (1, 0)\}$$

is a set of generators of  $\Gamma$ . In terms of these generators  $\Gamma$  has a presentation of the form

$$\Gamma = \langle a_i \mid a_x a_y a_z = 1 \quad \text{for } (x, y, z) \in \mathcal{T} \rangle$$

where  $\mathcal{T}$  is a so-called triangle presentation. Each element of  $\Gamma$  has a unique shortest expression of the form

$$x = a_{i_1} \cdots a_{i_m} a_{i_{m+1}}^{-1} \cdots a_{i_{m+n}}^{-1}$$

where the  $a_{i_j} \in N$  and the shape function  $\sigma$  on  $\Gamma$  is given by

$$\sigma(a_{i_1} \cdots a_{i_m} a_{i_{m+1}}^{-1} \cdots a_{i_{m+n}}^{-1}) = (m, n).$$

**COROLLARY 4.3.** *Let  $\Gamma$  be an  $\tilde{A}_2$  group and  $\rho$  be the right regular representation of  $\ell^1(\Gamma)$  on  $\ell^2(\Gamma)$  defined by  $\rho(g)f = f * g$ . If  $g \in \mathbb{C}\Gamma$  is supported on words of shape  $(m, n)$  then*

$$\|\rho(g)\| \leq 1/2(m+1)(n+1)(m+n+2)\sqrt{\max(m, n)+1} \|g\|_2.$$

Moreover,  $\Gamma$  has property (RD).

*Proof.* These results follow from Theorem 1.1 and Corollary 1.10. □

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