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POSETS AND DIFFERENTIAL GRADED ALGEBRAS

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Abstract

If P is a partially ordered set and R is a commutative ring, then a certain differential graded R-algebra $A_\bullet(P)$ is defined from the order relation on P. The algebra $A_\bullet(\emptyset)$ corresponding to the empty poset is always contained in $A_\bullet(P)$ so that $A_\bullet(P)$ can be regarded as an $A_\bullet(\emptyset)$-algebra. The main result of this paper shows that if R is an integral domain and P and P' are finite posets such that $A_\bullet(P) \cong A_\bullet(P')$ as differential graded $A_\bullet(\emptyset)$-algebras, then P and P' are isomorphic.

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1. Introduction

A common way to study partially ordered sets involves associating certain algebraic objects with a poset and then trying to gain new insights by considering these associated objects. For example, the concept of a Cohen–Macaulay poset arises naturally from the study of Stanley–Reisner rings [1, 3]. On the other hand, algebraic constructions associated with partially ordered sets have also proven to have widespread applicability within algebra itself, particularly in the area of representation theory [2].

The current work, which grew out of an interest in posets that arise in group representation theory, is based upon this interplay between partially ordered sets and algebra. If P is a partially ordered set and R is an integral domain, then we define a graded R-algebra $A_\bullet(P)$. The definition involves forming a new poset P_0 by adjoining a minimum element 0 to the poset P. For any $n \geq 0$ the component $A_n(P)$ of degree n is the free R-module on the symbols $[x_1 < \cdots < x_n]$ whenever $x_1 < \cdots < x_n$ is a chain in P_0. Using the order relation on P_0, one can define a multiplication on $A_\bullet(P)$,

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and it also has an R-endomorphism of degree -1 that makes $A_\bullet(P)$ into a differential graded R-algebra. The algebra $A_\bullet(\emptyset)$ corresponding to the empty poset is necessarily contained in $A_\bullet(P)$ so that $A_\bullet(P)$ is in fact an $A_\bullet(\emptyset)$-algebra.

Now suppose that P and P' are finite posets and $f_\bullet : A_\bullet(P) \to A_\bullet(P')$ is an isomorphism of differential graded $A_\bullet(\emptyset)$-algebras. If $f_\bullet$ maps the distinguished basis of $A_\bullet(P)$ to that of $A_\bullet(P')$, then the definition of the multiplication in $A_\bullet(P)$ makes it easy to see that P and P' are isomorphic. The main result of this paper shows that this conclusion is valid even if $f_\bullet$ does not preserve the distinguished basis. Thus one can recover the poset P from the algebra $A_\bullet(P)$ with no additional information.

Section 2 of the paper contains the definition of $A_\bullet(P)$ and a proof that it is a differential graded $A_\bullet(\emptyset)$-algebra.The proof that the algebra $A_\bullet(P)$ determines the poset P is given in Section 3. Finally, Section 4 gives a description of the graded center in terms of certain annihilators in $A_\bullet(P)$. Although we have chosen to assume throughout the paper that the coefficient ring R is an integral domain, it should be noted that this assumption is often not necessary. In particular, all of the results of Section 2 hold over an arbitrary commutative ring.

2. The definition and basic properties of the algebra

If P is a partially ordered set and R is an integral domain, then we will define a differential graded R-algebra $A_\bullet(P)$ from the poset P. The first step is to define a new poset P_0 in which the points consist of the points in P, together with one additional point called 0. The order $<$ on P_0 is given by taking $x < y$ in P_0 if either $x = 0$ and $y \in P$ or $x, y \in P$ and $x < y$ in P.

For each $n \geq 0$ the component $A_n(P)$ is defined to be the free R-module on the symbols $[x_1 < x_2 < \cdots < x_n]$ whenever $x_1 < x_2 < \cdots < x_n$ is a strictly increasing chain in P_0. For convenience we will also use the symbol $[x_1 < x_2 < \cdots < x_n]$ even when $x_1, x_2, \ldots, x_n$ do not form a strictly increasing chain in P_0, but in this case we set $[x_1 < x_2 < \cdots < x_n]$ equal to 0 in $A_n(P)$. Note that $A_0(P)$ is a free R-module of rank one, generated by the symbol $[\ ]$.

Define a multiplication on the (non-zero) basis elements of $A_\bullet(P)$ by setting

$$[x_1 < \cdots < x_m][y_1 < \cdots < y_n] = \begin{cases} [x_1 < \cdots < x_m < y_1 < \cdots < y_n] & \text{if } x_m < y_1 \\ (-1)^{m-1}[0 < x_1 < \cdots < x_{m-1} < y_1 < \cdots < y_n] & \\ +(-1)^m[0 < x_1 < \cdots < x_m < y_2 < \cdots < y_n] & \text{if } x_m \not< y_1, \end{cases}$$

and extend this multiplication to all of $A_\bullet(P)$ by linearity. In the proofs of the following propositions it is important to bear in mind that the equation defining this multiplication applies only to products of non-zero generators of $A_\bullet(P)$.

PROPOSITION 2.1. *Let P be a partially ordered set. Then $A_\bullet(P)$ is a graded associative algebra with* 1.

PROOF. The identity element of $A_\bullet(P)$ is given by [ ], and it is clear from the definition of the product that $A_m(P)A_n(P) = A_{m+n}(P)$. Thus it is only necessary to show that $A_\bullet(P)$ is associative.

Let $a, b, c \in A_\bullet(P)$ be homogeneous elements. We will prove that $(ab)c = a(bc)$ by induction on $\deg b$. The equality clearly holds if $\deg a = 0$, $\deg b = 0$, or $\deg c = 0$, so assume that $\deg b = 1$, $\deg a \geq 1$, and $\deg c \geq 1$. To prove that $(ab)c = a(bc)$, it suffices to consider the case in which a, b, and c are non-zero homogeneous generators. Suppose, then, that $a = [x_1 < \cdots < x_m]$, $b = [y_1]$, and $c = [z_1 < \cdots < z_p]$. If $x_m < y_1 < z_1$, then it is easy to see that $(ab)c = a(bc)$, so suppose that $x_m \not< y_1$ but $y_1 < z_1$. Then

$$\begin{aligned}
(ab)c &= \big([x_1 < \cdots < x_m][y_1]\big)[z_1 < \cdots < z_p] \\
&= (-1)^{m-1}[0 < x_1 < \cdots < x_{m-1} < y_1][z_1 < \cdots < z_p] \\
&\quad + (-1)^m[0 < x_1 < \cdots < x_m][z_1 < \cdots < z_p] \\
&= (-1)^{m-1}[0 < x_1 < \cdots < x_{m-1} < y_1 < z_1 < \cdots < z_p] \\
&\quad + (-1)^m[0 < x_1 < \cdots < x_m < z_1 < \cdots < z_p] \\
&= [x_1 < \cdots < x_m][y_1 < z_1 < \cdots < z_p] \\
&= [x_1 < \cdots < x_m]\big([y_1][z_1 < \cdots < z_p]\big) \\
&= a(bc).
\end{aligned}$$

Similar computations show that $(ab)c = a(bc)$ when $x_m < y_1$ and $y_1 \not< z_1$, and also when $x_m \not< y_1$ and $y_1 \not< z_1$.

It follows that if a, b, and c are any homogeneous elements of $A_\bullet(P)$ with $\deg b = 1$, then $(ab)c = a(bc)$. Assume by induction that $n \geq 1$ and that if a, b, and c are homogeneous with $\deg b \leq n$, then $(ab)c = a(bc)$. Then

$$\begin{aligned}
\big(a[y_1 < \cdots < y_{n+1}]\big)c &= \big(a([y_1 < \cdots < y_n][y_{n+1}])\big)c \\
&= \big((a[y_1 < \cdots < y_n])[y_{n+1}]\big)c \\
&= \big(a[y_1 < \cdots < y_n]\big)\big([y_{n+1}]c\big) \\
&= a\big([y_1 < \cdots < y_n]([y_{n+1}]c)\big) \\
&= a\big(([y_1 < \cdots < y_n][y_{n+1}])c\big) \\
&= a\big([y_1 < \cdots < y_{n+1}]c\big).
\end{aligned}$$

Hence $(ab)c = a(bc)$ whenever a, b, and c are homogeneous with $\deg b \leq n + 1$, and it follows that $A_\bullet(P)$ is associative. This completes the proof.

If $1 \le i \le n$, then we write $[x_1 < \cdots < \hat{x}_i < \cdots < x_n]$ for $[x_1 < \cdots < x_{i-1} < x_{i+1} < \cdots < x_n]$. Define a sequence of R-linear maps $d : A_n(P) \to A_{n-1}(P)$ by setting

$$d[x_1 < \cdots < x_n] = \sum_{i=1}^{n} (-1)^{i-1} [x_1 < \cdots < \hat{x}_i < \cdots < x_n]$$

on all non-zero homogeneous generators $[x_1 < \cdots < x_n]$. It is easy to verify that $d^2 = 0$.

PROPOSITION 2.2. *Let P be a partially ordered set, and suppose that $a \in A_m(P)$ and $b \in A_n(P)$. Then*

$$d(ab) = (da)b + (-1)^m a(db),$$

and $(A_\bullet(P), d)$ is a differential graded R-algebra.

PROOF. We will prove that $d(ab) = (da)b + (-1)^m a(db)$ by induction on m. It is clear that the equation holds if $m = 0$ or $n = 0$, so assume that $m = 1$ and $n \ge 1$. To prove that the equation holds in this case, it suffices to consider the situation in which a and b are non-zero homogeneous generators. Suppose, then, that $a = [x_1]$ and $b = [y_1 < \cdots < y_n]$. If $x_1 < y_1$, then

$$\begin{aligned} &(da)b + (-1)^m a(db) \\ &\quad = [y_1 < \cdots < y_n] - \sum_{i=1}^{n} (-1)^{i-1} [x_1 < y_1 < \cdots < \hat{y}_i < \cdots < y_n] \\ &\quad = d[x_1 < y_1 < \cdots < y_n] = d(ab). \end{aligned}$$

Now suppose that $x_1 \not< y_1$. Then one can check that

$$\begin{aligned} &(da)b + (-1)^m a(db) \\ &\quad = [y_1 < \cdots < y_n] - \sum_{i=1}^{n} (-1)^{i-1} [x_1][y_1 < \cdots < \hat{y}_i < \cdots < y_n] \\ &\quad = [y_1 < \cdots < y_n] - [x_1][y_2 < \cdots < y_n] \\ &\qquad - \sum_{i=2}^{n} \big((-1)^{i-1} [0 < y_1 < \cdots < \hat{y}_i < \cdots < y_n] \\ &\qquad\qquad + (-1)^i [0 < x_1 < y_2 < \cdots < \hat{y}_i < \cdots < y_n] \big) \end{aligned}$$

$$\begin{aligned}
&= [y_1 < \cdots < y_n] + \sum_{i=1}^{n}(-1)^i[0 < y_1 < \cdots < \hat{y}_i < \cdots < y_n] \\
&\quad - [x_1][y_2 < \cdots < y_n] + [0 < y_2 < \cdots < y_n] \\
&\quad - \sum_{i=2}^{n}(-1)^i[0 < x_1 < y_2 < \cdots < \hat{y}_i < \cdots < y_n] \\
&= d[0 < y_1 < \cdots < y_n] - d[0 < x_1 < y_2 < \cdots < y_n] \\
&= d\big([x_1][y_1 < \cdots < y_n]\big) = d(ab).
\end{aligned}$$

It now follows that if a and b are any homogeneous elements of $A_\bullet(P)$ with $\deg a = 1$, then $d(ab) = (da)b - a(db)$. Assume by induction that $m \geq 1$ and that if a and b are homogeneous with $\deg a \leq m$, then $d(ab) = (da)b + (-1)^{\deg a}a(db)$. Then

$$\begin{aligned}
&\big(d[x_1 < \cdots < x_{m+1}]\big)b + (-1)^{m+1}[x_1 < \cdots < x_{m+1}]db \\
&\quad = d\big([x_1][x_2 < \cdots < x_{m+1}]\big)b + (-1)^{m+1}[x_1 < \cdots < x_{m+1}]db \\
&\quad = [x_2 < \cdots < x_{m+1}]b - [x_1]\big(d[x_2 < \cdots < x_{m+1}]\big)b \\
&\quad\quad + (-1)^{m+1}[x_1 < \cdots < x_{m+1}]db \\
&\quad = [x_2 < \cdots < x_{m+1}]b \\
&\quad\quad - [x_1]\big((d[x_2 < \cdots < x_{m+1}])b + (-1)^m[x_2 < \cdots < x_{m+1}]db\big) \\
&\quad = (d[x_1])[x_2 < \cdots < x_{m+1}]b - [x_1]d\big([x_2 < \cdots < x_{m+1}]b\big) \\
&\quad = d\big([x_1 < x_2 < \cdots < x_{m+1}]b\big).
\end{aligned}$$

Hence $d(ab) = (da)b + (-1)^{\deg a}a(db)$ whenever a and b are homogeneous with $\deg a \leq m + 1$, and it follows that $A_\bullet(P)$ is a differential graded R-algebra.

If P is any poset, then the algebra $A_\bullet(\emptyset)$ corresponding to the empty poset is just the subalgebra of $A_\bullet(P)$ spanned by [ ] and [0]. Thus $A_\bullet(P)$ is actually a differential graded $A_\bullet(\emptyset)$-algebra. Unless otherwise specified, therefore, any homomorphism $g_\bullet : A_\bullet(P) \to A_\bullet(P')$ that we consider will be assumed to be a homomorphism of differential graded $A_\bullet(\emptyset)$-algebras so that $g_\bullet([0]) = [0]$. For simplicity of notation we generally write $g_\bullet[x_1 < \cdots < x_n]$ instead of $g_\bullet([x_1 < \cdots < x_n])$.

Let P and P' be partially ordered sets, and let $f_1 : A_1(P) \to A_1(P')$ be an R-linear map given by

$$f_1[x] = \sum\nolimits_{x' \in P'_0} c_{x'x}[x']$$

for some elements $c_{x'x} \in R$. We want to explore the conditions under which f_1 extends to a homomorphism $f_\bullet : A_\bullet(P) \to A_\bullet(P')$ of differential graded $A_\bullet(\emptyset)$-algebras. The matrix $C = (c_{x'x})$ will be referred to as the *matrix of* f_1.

Let $f_0 : A_0(P) \to A_0(P')$ be the unique R-linear map satisfying $f_0[\ ] = [\ ]$, and for $n \geq 2$ let $f_n : A_n(P) \to A_n(P')$ be the unique R-linear map defined on basis elements of $A_n(P)$ by

$$f_n[y_1 < \cdots < y_n] = f_1[y_1] \cdots f_1[y_n].$$

In this way we associate an R-linear map $f_\bullet : A_\bullet(P) \to A_\bullet(P')$ to each R-linear map $f_1 : A_1(P) \to A_1(P')$.

LEMMA 2.3. *Let P and P' be posets, and let $f_1 : A_1(P) \to A_1(P')$ be an R-linear map with matrix $C = (c_{x'x})$. Then the R-linear map $f_\bullet : A_\bullet(P) \to A_\bullet(P')$ satisfies $df_1 = f_0 d$ if and only if $\sum_{x' \in P'_0} c_{x'x} = 1$ for all $x \in P_0$.*

PROOF. Let $x \in P_0$. Then $df_1[x] = d \sum_{x' \in P'_0} c_{x'x}[x'] = \sum_{x' \in P'_0} c_{x'x}[\ ]$, and $f_0 d[x] = f_0[\ ] = [\ ]$. Hence $df_1[x] = f_0 d[x]$ if and only if $\sum_{x' \in P'_0} c_{x'x} = 1$, as desired.

LEMMA 2.4. *Let P and P' be posets, and let $f_1 : A_1(P) \to A_1(P')$ be an R-linear map with matrix $C = (c_{x'x})$. Suppose that $f_1[0] = [0]$ and that $df_1 = f_0 d$. Then the following conditions are equivalent:*

(1) *If $x, y \in P_0$ and $x \not< y$, then $[0]f_1[x]f_1[y] = 0$.*
(2) *If $a, b \in A_\bullet(P)$, then $f_\bullet(ab) = f_\bullet(a)f_\bullet(b)$.*
(3) *If $x \not< y$ in P_0 and $0 \neq x' < y'$ in P'_0, then $c_{x'x}c_{y'y} = 0$.*

PROOF. Let $x, y \in P_0$ with $x \not< y$. Then

$$[0]f_1[x]f_1[y] = [0] \sum_{x' \in P'_0} c_{x'x}[x'] \sum_{y' \in P'_0} c_{y'y}[y'] = \sum_{0 \neq x' < y'} c_{x'x}c_{y'y}[0 < x' < y'],$$

and it follows that (1) and (3) are equivalent.

Now suppose that (2) holds. If $x, y \in P_0$ and $x \not< y$, then

$$[0]f_1[x]f_1[y] = f_3\big([0][x][y]\big) = f_3\big([0][0 < y] - [0][0 < x]\big) = 0.$$

Thus we see that (2) implies (1).

Finally, we show that (3) implies (2). To prove that $f_\bullet(ab) = f_\bullet(a)f_\bullet(b)$ for all $a, b \in A_\bullet(P)$, it suffices to consider the case in which a and b are homogeneous basis elements. In fact, it is enough to prove that

$$f_{n+1}\big([x][y_1 < \cdots < y_n]\big) = f_1[x]f_n[y_1 < \cdots < y_n]$$

whenever $x \in P_0$ and $y_1 < \cdots < y_n$ in P_0. The result is immediate if $n = 0$, so assume that $n \geq 1$. If $x < y_1$, then

$$\begin{aligned} f_{n+1}\big([x][y_1 < \cdots < y_n]\big) &= f_{n+1}[x < y_1 < \cdots < y_n] \\ &= f_1[x]f_1[y_1]\cdots f_1[y_n] \\ &= f_1[x]f_n[y_1 < \cdots < y_n], \end{aligned}$$

as desired. Thus we may assume that $x \not< y_1$.

We now prove that if $n \geq 1$ and $x \not< y_1$, then $f_{n+1}\big([x][y_1 < \cdots < y_n]\big) = f_1[x]f_n[y_1 < \cdots < y_n]$. First suppose that $n = 1$. Then (3) implies that

$$\begin{aligned} f_1[x]f_1[y_1] &= \sum_{x',y' \in P_0'} c_{x'x}c_{y'y_1}[x'][y'] \\ &= \sum_{y' \in P'} \sum_{0 \neq x' < y'} c_{x'x}c_{y'y_1}[x' < y'] + \sum_{y' \in P'} c_{0x}c_{y'y_1}[0 < y'] \\ &\quad - \sum_{x' \in P'} c_{x'x}c_{0y_1}[0 < x'] + \sum_{y' \in P'} \sum_{0 \neq x' \not< y'} c_{x'x}c_{y'y_1}\big([0 < y'] - [0 < x']\big) \\ &= \sum_{y' \in P'} \Bigg(c_{0x}c_{y'y_1} - c_{y'x}c_{0y_1} + \sum_{0 \neq x' \not< y'} c_{x'x}c_{y'y_1} - \sum_{0 \neq x' \not> y'} c_{y'x}c_{x'y_1}\Bigg)[0 < y'] \\ &= \sum_{y' \in P'} \Bigg(\sum_{x' \in P_0'} c_{x'x}c_{y'y_1} - \sum_{x' \in P_0'} c_{y'x}c_{x'y_1}\Bigg)[0 < y']. \end{aligned}$$

Since $df_1 = f_0 d$, Lemma 2.3 implies that

$$\begin{aligned} f_1[x]f_1[y_1] &= \sum_{y' \in P'} (c_{y'y_1} - c_{y'x})[0 < y'] \\ &= \sum_{y' \in P_0'} c_{y'y_1}[0][y'] - \sum_{y' \in P_0'} c_{y'x}[0][y'] \\ &= [0]f_1[y_1] - [0]f_1[x] \\ &= f_2[0 < y_1] - f_2[0 < x] \\ &= f_2\big([x][y_1]\big). \end{aligned} \tag{2.5}$$

Now suppose that $n \geq 2$. Using (2.5) and (1), we see that

$$\begin{aligned} f_1[x]f_n[y_1 < \cdots < y_n] &= f_1[x]f_1[y_1]\cdots f_1[y_n] \\ &= [0]f_1[y_1]\cdots f_1[y_n] - [0]f_1[x]f_1[y_2]\cdots f_1[y_n] \\ &= f_{n+1}[0 < y_1 < \cdots < y_n] - f_{n+1}[0 < x < y_2 < \cdots < y_n] \\ &= f_{n+1}\big([x][y_1 < \cdots < y_n]\big). \end{aligned}$$

Thus (2) follows, and this completes the proof.

PROPOSITION 2.6. *Let P and P' be partially ordered sets, and let $f_1 : A_1(P) \to A_1(P')$ be an R-linear map with matrix $C = (c_{x'x})$. Then f_1 extends to a homomorphism $f_\bullet : A_\bullet(P) \to A_\bullet(P')$ of differential graded $A_\bullet(\emptyset)$-algebras if and only if the following conditions are satisfied.*

(1) $c_{00} = 1$ *and* $c_{x'0} = 0$ *for all* $x' \in P'$.
(2) $\sum_{x' \in P'_0} c_{x'x} = 1$ *for all* $x \in P_0$.
(3) *If* $x \not< y$ *in* P_0 *and* $0 \neq x' < y'$ *in* P'_0, *then* $c_{x'x}c_{y'y} = 0$.

PROOF. Note that f_1 extends to a homomorphism $f_\bullet$ of differential graded $A_\bullet(\emptyset)$-algebras if and only if the following conditions are satisfied:

(1′) $f_0[\ ] = [\ ]$ and $f_1[0] = [0]$.
(2′) $df_{n+1} = f_n d$ for all $n \geq 0$.
(3′) $f_\bullet(ab) = f_\bullet(a)f_\bullet(b)$ for all $a, b \in A_\bullet(P)$.

Thus it suffices to show that conditions (1), (2), and (3) are equivalent to conditions (1′), (2′), and (3′). We have defined f_0 so that $f_0[\ ] = [\ ]$, and $f_1[0] = [0]$ precisely when $c_{00} = 1$ and $c_{x'0} = 0$ for all $x' \in P'$. Thus (1) is equivalent to (1′).

Suppose that (1′), (2′), and (3′) hold. Then Lemma 2.3 implies that (2) holds, and Lemma 2.4 implies that (3) holds.

Conversely, suppose that f_1 satisfies (1), (2), and (3). Then $f_\bullet$ also satisfies (1′), and Lemma 2.3 implies that $df_1 = f_0 d$. By Lemma 2.4 it follows that $f_\bullet$ satisfies (3′), so it only remains to show that $df_{n+1} = f_n d$ for $n \geq 1$. If $[y_1 < \cdots < y_{n+1}]$ is any basis element of $A_{n+1}(P)$, then by induction it follows that

$$\begin{aligned} df_{n+1}&[y_1 < \cdots < y_{n+1}] \\ &= d\big(f_n[y_1 < \cdots < y_n]f_1[y_{n+1}]\big) \\ &= \big(df_n[y_1 < \cdots < y_n]\big)f_1[y_{n+1}] + (-1)^n f_n[y_1 < \cdots < y_n]df_1[y_{n+1}] \\ &= \big(f_{n-1}d[y_1 < \cdots < y_n]\big)f_1[y_{n+1}] + (-1)^n f_n[y_1 < \cdots < y_n]f_0 d[y_{n+1}] \\ &= f_n\big((d[y_1 < \cdots < y_n])[y_{n+1}] + (-1)^n[y_1 < \cdots < y_n]d[y_{n+1}]\big) \\ &= f_n d[y_1 < \cdots < y_{n+1}]. \end{aligned}$$

This completes the proof.

COROLLARY 2.7. *Let $f : P \to P'$ be a map of posets. Then the following conditions are equivalent.*

(1) *There is a homomorphism $f_\bullet : A_\bullet(P) \to A_\bullet(P')$ of differential graded $A_\bullet(\emptyset)$-algebras satisfying $f_1[x] = \big[f(x)\big]$ for all $x \in P$.*

(2) *There is a homomorphism $f_\bullet : A_\bullet(P) \to A_\bullet(P')$ of differential graded $A_\bullet(\emptyset)$-algebras such that f_n satisfies*

$$f_n[x_1 < \cdots < x_n] = \big[f(x_1) < \cdots < f(x_n)\big] \quad \textit{for all } n \geq 1.$$

(3) *If* $f(x) < f(y)$, *then* $x < y$ *for all* $x, y \in P$.

PROOF. First suppose that (1) holds. We will prove by induction on n that f_n is given by

$$f_n[x_1 < \cdots < x_n] = \big[f(x_1) < \cdots < f(x_n)\big]$$

for all $n \geq 1$. This equation is true for $n = 1$ by assumption. Let $[x_1 < \cdots < x_{n+1}]$ be a non-zero homogeneous generator. Because $x_n < x_{n+1}$ and f is a map of posets, it follows that $f(x_n) \leq f(x_{n+1})$. Thus

$$\big[f(x_1) < \cdots < f(x_n)\big]\big[f(x_{n+1})\big] = \big[f(x_1) < \cdots < f(x_{n+1})\big]$$

even if $f(x_n) = f(x_{n+1})$. Hence

$$\begin{aligned} f_{n+1}[x_1 < \cdots < x_{n+1}] &= f_{n+1}\big([x_1 < \cdots < x_n][x_{n+1}]\big) \\ &= f_n[x_1 < \cdots < x_n] f_1[x_{n+1}] \\ &= \big[f(x_1) < \cdots < f(x_n)\big]\big[f(x_{n+1})\big] \\ &= \big[f(x_1) < \cdots < f(x_{n+1})\big], \end{aligned}$$

and (2) follows.

It is trivial that (2) implies (1), so assume that (1) holds. If $x \in P$, then the matrix $C = (c_{x'x})$ of f_1 satisfies $c_{x'x} = 1$ if $x' = f(x)$ and $c_{x'x} = 0$ if $x' \neq f(x)$. Proposition 2.6 shows that if $x \nless y$ in P_0 and $0 \neq x' < y'$ in P'_0, then $c_{x'x}c_{y'y} = 0$. But if $x, y \in P$ are elements such that $f(x) < f(y)$, then $c_{f(x),x}c_{f(y),y} = 1$, so it follows that $x < y$. Hence (1) implies (3).

Finally, suppose that (3) holds. Extend f to a map $f : P_0 \to P'_0$ by defining $f(0) = 0$, and let $f_1 : A_1(P) \to A_1(P')$ be the R-linear map satisfying $f_1[x] = \big[f(x)\big]$ for all $x \in P_0$. Then all of the conditions of Proposition 2.6 are satisfied, and it follows that f_1 extends to a homomorphism $f_\bullet : A_\bullet(P) \to A_\bullet(P')$ of differential graded $A_\bullet(\emptyset)$-algebras, as desired.

Finally, we end this section with the following simple but useful observation.

PROPOSITION 2.8. *If* P *is a poset, then* $A_\bullet(P)$ *is contractible. In fact, if* $s_\bullet : A_\bullet(P) \to A_\bullet(P)$ *is the map of degree one satisfying* $s_\bullet(x) = [0]x$ *for every homogeneous element* $x \in A_\bullet(P)$, *then* $s_\bullet$ *is a contracting homotopy.*

PROOF. Let $x \in A_\bullet(P)$ be homogeneous. Because d is a derivation, it follows that $ds_\bullet(x) + s_\bullet d(x) = d\big([0]x\big) + [0](dx) = x$, as desired.

3. Isomorphic algebras

In this section our goal is to show that if P and P' are finite posets such that $A_\bullet(P) \cong A_\bullet(P')$ as differential graded $A_\bullet(\emptyset)$-algebras, then $P \cong P'$. While this fact is obvious if there is an isomorphism from $A_\bullet(P)$ to $A_\bullet(P')$ that maps each basis element $[x_1 < \cdots < x_n]$ of $A_\bullet(P)$ to a basis element of $A_\bullet(P')$, not all isomorphisms arise in this way. Nevertheless, it is easy to see that certain invariants associated with the posets are the same. For example, the rank of $A_1(P)$ is just the cardinality $|P_0| = |P| + 1$, so it follows that $|P| = |P'|$.

Another invariant that can easily be recovered from the algebra $A_\bullet(P)$ is the height of the poset. Recall that an element $x \in P$ is said to have *height* $h_P(x) = n$ if n is the largest number such that there is a chain $x_1 < \cdots < x_n = x$ in P. The *height* $h(P)$ *of the poset* P is defined to be the supremum of the heights of its elements. If P is finite with $h(P) = n$, then $h(P_0) = n + 1$ so that $A_{n+1}(P) \neq 0$ but $A_m(P) = 0$ for all $m > n+1$. Thus $h(P) = h(P')$ if P and P' are finite posets such that $A_\bullet(P) \cong A_\bullet(P')$. A connection between $A_\bullet(P)$ and the heights of individual elements in P is given by the following lemma.

LEMMA 3.1. *Let P be a poset, and let $x \in P$. If there is an element $a \in A_{n-1}(P)$ such that $[0]a[x] \neq 0$, then $h_P(x) \geq n$.*

PROOF. It suffices to consider the case in which $n \geq 2$. Suppose that $a \in A_{n-1}(P)$ is an element such that $[0]a[x] \neq 0$. Then there is a basis element $[y_1 < \cdots < y_{n-1}] \in A_{n-1}(P)$ such that $[0][y_1 < \cdots < y_{n-1}][x] \neq 0$, so the product $[y_1 < \cdots < y_{n-1}][x]$ does not lie in the ideal $[0]A_\bullet(P)$ generated by $[0]$. Hence $y_1 \neq 0$ and $y_{n-1} < x$ so that $y_1 < \cdots < y_{n-1} < x$ is a chain in P. Thus $h_P(x) \geq n$, as desired.

PROPOSITION 3.2. *Suppose that P and P' are finite posets and $f_\bullet : A_\bullet(P) \to A_\bullet(P')$ is an isomorphism such that $C = (c_{x'x})$ is the matrix of f_1. Let $H \subseteq P$ and $H' \subseteq P'$ be the subposets consisting of all elements that are not of maximum height, and let $x' \in P'$. Then $x' \in H'$ if and only if $c_{x'x} \neq 0$ for some $x \in H$.*

PROOF. Suppose that $x' \in P'$ is an element such that $c_{x'x} = 0$ for all $x \in H$. Because $f_\bullet$ is an isomorphism, there are distinct elements $m_1, \ldots, m_s \in P - H$ and $b_1, \ldots, b_s \in R - \{0\}$ such that $[x'] = b_1 f_1[m_1] + \cdots + b_s f_1[m_s]$. Let $0 < x_1 < \cdots < x_{n-1} < m_1$ be a chain of maximum length in P_0, and set $a = b_1[m_1] + \cdots + b_s[m_s] \in A_1(P)$. Then $[0 < x_1 < \cdots < x_{n-1}]a \neq 0$, so

$$0 \neq f_{n+1}\big([0 < x_1 < \cdots < x_{n-1}]a\big) = [0]f_{n-1}[x_1 < \cdots < x_{n-1}][x'].$$

It follows by Lemma 3.1 that

$$h_{P'}(x') \geq n = h_P(m_1) = h(P) = h(P').$$

Hence $x' \notin H'$, as desired.

Conversely, suppose that $x' \in P' - H'$ and $x \in P$ are elements such that $c_{x'x} \neq 0$. Let $0 < x'_1 < \cdots < x'_{n-1} < x'$ be a chain of maximum length in P'_0, and let $b \in A_{n-1}(P)$ be the element with $f_{n-1}(b) = [x'_1 < \cdots < x'_{n-1}]$. Then

$$f_{n+1}\big([0]b[x]\big) = [0 < x'_1 < \cdots < x'_{n-1}] \sum\nolimits_{y' \in P'_0} c_{y'x}[y']$$

is non-zero because $c_{x'x}[0 < x'_1 < \cdots < x'_{n-1} < x'] \neq 0$. Hence $[0]b[x] \neq 0$, and Lemma 3.1 implies that

$$h_P(x) \geq n = h_{P'}(x') = h(P') = h(P).$$

Thus $x \notin H$, and this completes the proof.

COROLLARY 3.3. *Suppose that P and P' are finite posets and $f_\bullet : A_\bullet(P) \to A_\bullet(P')$ is an isomorphism. Let $H \subseteq P$ and $H' \subseteq P'$ be the subposets consisting of all elements that are not of maximum height. Then $f_\bullet$ restricts to an isomorphism $h_\bullet : A_\bullet(H) \to A_\bullet(H')$.*

PROPOSITION 3.4. *Let P and P' be finite posets, and let $f_\bullet : A_\bullet(P) \to A_\bullet(P')$ be an isomorphism such that $C = (c_{x'x})$ is the matrix of f_1. If $x \in P$ and $x' \in P'$ are elements with $c_{x'x} \neq 0$, then $h_{P'}(x') \leq h_P(x)$.*

PROOF. The proof proceeds by induction on $h(P)$. The result is obvious if $h(P) = 1$, so assume that $h(P) > 1$. Let $H \subseteq P$ and $H' \subseteq P'$ be the subposets consisting of all elements that are not of maximum height. Corollary 3.3 implies that if $x \in H$ and $x' \in P'$ are elements such that $c_{x'x} \neq 0$, then $x' \in H'$. Then $h_{H'}(x') \leq h_H(x)$ by induction, and the result follows in this case. On the other hand, if $x \in P - H$, then

$$h_P(x) = h(P) = h(P') \geq h_{P'}(x')$$

for all $x' \in P'$, as desired.

DEFINITION 3.5. Let P be a finite poset, and let $a \in A_1(P)$. Write $a = \sum_{x \in P_0} a_x[x]$. The set supp $a = \{x \in P \mid a_x \neq 0\}$ will be called the *support* of a in P.

Let P' be another poset, and let $f_\bullet : A_\bullet(P) \to A_\bullet(P')$ be an $A_\bullet(\emptyset)$-isomorphism. Two elements $x \in P$ and $x' \in P'$ will be called *mutually $f_\bullet$-supportive* (or simply *mutually supportive* when $f_\bullet$ is understood) provided that $x' \in$ supp $f_1[x]$ and $x \in$ supp $f_1^{-1}[x']$.

Note that the support of an element $a \in A_1(P)$ is defined to be a subset of P, not of P_0; we do not consider 0 to lie in the support of a even if $a_0 \neq 0$.

It will be important to observe that if $f_\bullet : A_\bullet(P) \to A_\bullet(P')$ is an isomorphism and $x \in P$, then there is always an element $x' \in P'$ such that x and x' are mutually supportive. Indeed, suppose that C is the matrix of f_1 and D is the matrix of f_1^{-1}. Then $1 = \sum_{x' \in P_0'} d_{xx'} c_{x'x}$, and there is an element $x' \in P_0'$ such that $d_{xx'} c_{x'x} \neq 0$. Because f_1 is an isomorphism with $f_1[0] = [0]$, it is easy to see that $x' \neq 0$. Then $x \in P$ and $x' \in P'$ are mutually supportive. Moreover, any two mutually supportive elements must have the same height by Proposition 3.4.

If P is a finite partially ordered set, then it will sometimes be useful to consider total orders on P_0 in addition to the original partial order. For convenience we will generally specify a total ordering on P_0 simply by listing all of the elements $x_0, \ldots, x_n$ of P_0 in increasing order. The symbol $<$ will still be reserved for the partial order on P_0.

DEFINITION 3.6. Let P be a partially ordered set with $|P| = n$, and write $P_0 = \{x_0, x_1, \ldots, x_n\}$. We will say that $x_0, x_1, \ldots, x_n$ is a *tall order* on P_0 if $i < j$ whenever $h_{P_0}(x_i) < h_{P_0}(x_j)$.

Suppose that $x_0, x_1, \ldots, x_n$ is a tall order on P_0, and suppose that $x_i < x_j$ for some i and j. Then $h_{P_0}(x_i) < h_{P_0}(x_j)$, so $i < j$. Thus the total ordering on P_0 specified by $x_0, x_1, \ldots, x_n$ is compatible with the original partial ordering. In particular, $x_0 = 0$.

Now suppose that P and P' are finite partially ordered sets, and let $f_\bullet : A_\bullet(P) \to A_\bullet(P')$ be an $A_\bullet(\emptyset)$-isomorphism. Suppose that $x_0, \ldots, x_n$ is a tall order on P_0 and $x_0', \ldots, x_n'$ is a tall order on P_0'. If C is the matrix of f_1, then for simplicity write c_{ij} for $c_{x_i' x_j}$. For any integer m with $1 \leq m \leq n$ let $P(m)$ be the subposet of P given by $P(m) = \{x_1, \ldots, x_m\}$, and let $P'(m)$ be the subposet of P' given by $P'(m) = \{x_1', \ldots, x_m'\}$. Let $f_1^{(m)} : A_1(P(m)) \to A_1(P'(m))$ be the R-linear map satisfying

$$f_1^{(m)}[x_i] = \left(1 - \sum_{j=1}^{m} c_{ji}\right)[0] + \sum_{j=1}^{m} c_{ji}[x_j']$$

for $0 \leq i \leq m$. Then Proposition 2.6 shows that $f_1^{(m)}$ extends to a homomorphism $f_\bullet^{(m)} : A_\bullet(P(m)) \to A_\bullet(P'(m))$ of differential graded $A_\bullet(\emptyset)$-algebras. We will say that the orderings $x_0, \ldots, x_n$ of P_0 and $x_0', \ldots, x_n'$ of P_0' are *$f_\bullet$-compatible* if $f_\bullet^{(m)}$ is an isomorphism such that x_m and x_m' are mutually $f_\bullet^{(m)}$-supportive for $1 \leq m \leq n$. Note that this condition implies that $x_m' \in \operatorname{supp} f_1[x_m]$ for all m.

PROPOSITION 3.7. *Assume that R is a field. Let P and P' be finite posets of height one, and let $f_\bullet : A_\bullet(P) \to A_\bullet(P')$ be an $A_\bullet(\emptyset)$-isomorphism. Let $0 = x_0, x_1, \ldots, x_n$ be any ordering of P_0. Then there exists an $f_\bullet$-compatible ordering $x_0', \ldots, x_n'$ of P_0'.*

PROOF. The proof proceeds by induction on $n = |P|$. If $n = 1$, then $P = \{x_1\}$. Let $x'_0 = 0$, and let x'_1 be the unique element of P'. Because x_1 and x'_1 must be mutually $f_\bullet$-supportive, the orderings x_0, x_1 and x'_0, x'_1 are $f_\bullet$-compatible.

Now suppose that $n > 1$. Let $x = x_n \in P$, and let $x' \in P'$ be an element such that x and x' are mutually $f_\bullet$-supportive. Let C be the matrix of f_1, and let D be the matrix of f_1^{-1} so that $c_{x'x} \neq 0$ and $d_{xx'} \neq 0$. Set $Q = P - \{x\}$ and $Q' = P' - \{x'\}$, and let $g_1 : A_1(Q) \to A_1(Q')$ be the R-linear map satisfying

$$g_1[y] = (c_{0y} + c_{x'y})[0] + \sum_{y' \in Q'} c_{y'y}[y']$$

for all $y \in Q_0$. By Proposition 2.6 the map g_1 extends to an $A_\bullet(\emptyset)$-homomorphism $g_\bullet : A_\bullet(Q) \to A_\bullet(Q')$, and we will show that $g_\bullet$ is an isomorphism.

Let B be the matrix of g_1, and let B_0 be the submatrix obtained by deleting the row and column corresponding to the basis element $[0]$. Because $g_1[0] = [0]$, expanding by minors along the column corresponding to $[0]$ shows that $\det B = \det B_0$. But B_0 is also the submatrix of C obtained by deleting the rows corresponding to $[0]$ and $[x']$ and the columns corresponding to $[0]$ and $[x]$. Because $D = C^{-1}$ and $f_1[0] = [0]$, it follows that $d_{xx'} = \det B_0 / \det C$. But $d_{xx'} \neq 0$, so $\det B = \det B_0 \neq 0$. Hence $g_\bullet$ is an isomorphism.

It now follows by induction that there exists an ordering $x'_0, \dots, x'_{n-1}$ of Q'_0 that is $g_\bullet$-compatible with the ordering $x_0, \dots, x_{n-1}$ of Q_0. Set $x'_n = x'$. Because $g_\bullet = f_\bullet^{(n-1)}$, the orderings $x_0, \dots, x_n$ of P_0 and $x'_0, \dots, x'_n$ of P'_0 are $f_\bullet$-compatible. This completes the proof.

The next result is essentially a convenient restatement of Proposition 2.6(3).

LEMMA 3.8. *Suppose that P and P' are finite posets and $f_\bullet : A_\bullet(P) \to A_\bullet(P')$ is an $A_\bullet(\emptyset)$-isomorphism. Let $x, y \in P$ and $x', y' \in P'$ be elements such that $x' \in \operatorname{supp} f_1[x]$ and $y' \in \operatorname{supp} f_1[y]$. If $x' < y'$, then $x < y$.*

PROOF. Let C be the matrix of f_1. Then $c_{x'x} \neq 0$ and $c_{y'y} \neq 0$, so $c_{x'x}c_{y'y} \neq 0$. If $x' < y'$, then Proposition 2.6(3) implies that $x < y$.

Suppose that P is a poset, S is a subset of P, and $y \in P$. We will write $S < y$ if $x < y$ for all $x \in S$. Recall that $P_{<y}$ denotes the subposet of P consisting of all elements x such that $x < y$. Thus $S < y$ if and only if $S \subseteq P_{<y}$.

LEMMA 3.9. *Assume that P and P' are finite posets and $f_\bullet : A_\bullet(P) \to A_\bullet(P')$ is an isomorphism. Let $H \subseteq P$ and $H' \subseteq P'$ be the subposets consisting of all elements that are not of maximum height, and let $h_\bullet : A_\bullet(H) \to A_\bullet(H')$ be the isomorphism* *obtained by restricting $f_\bullet$ to $A_\bullet(H)$. Suppose that there exist isomorphisms of posets $\psi : H \to H'$ and $\psi' : H' \to H$ and tall orders $x_0, \dots, x_m$ on H_0 and $x'_0, \dots, x'_m$ on H'_0 such that $x_0, \dots, x_m$ is $h_\bullet$-compatible with $0, \psi(x_1), \dots, \psi(x_m)$ and $x'_0, \dots, x'_m$ is $h_\bullet^{-1}$-compatible with $0, \psi'(x'_1), \dots, \psi'(x'_m)$. If $S \subseteq H$, let $e(S)$ denote the number of $y \in P - H$ such that $S = P_{<y}$; if $S' \subseteq H'$, let $e'(S')$ denote the number of $y' \in P' - H'$ such that $S' = P'_{<y'}$. Then $e(S) = e'(\psi(S))$ for all $S \subseteq H$, and $e'(S') = e(\psi'(S'))$ for all $S' \subseteq H'$.*

PROOF. If $S \subseteq H$, let $g(S)$ denote the number of elements $y \in P - H$ such that $S < y$; define $g'(S')$ similarly for any $S' \subseteq H'$.

Fix $S \subseteq H$, and suppose that there is an element $y' \in P' - H'$ such that $\psi(S) < y'$. Let y be an element of P such that $y' \in \operatorname{supp} f_1[y]$. Then $y \in P - H$ by Proposition 3.4. Let x be an element of S, and let i be the index such that $x = x_i$. Then x_i and $\psi(x_i)$ are mutually $h_\bullet^{(i)}$-supportive, and the definition of $h_1^{(i)}$ shows that $\psi(x_i) \in \operatorname{supp} f_1[x_i]$. But $\psi(x_i) < y'$, so Lemma 3.8 implies that $x = x_i < y$ and hence $S < y$. Because this holds for every y such that $y' \in \operatorname{supp} f_1[y]$, the element $a \in A_1(P)$ such that $f_1(a) = [y']$ is an R-linear combination of an element of $A_1(H)$ and elements $[y]$ such that $S < y$. It follows that $g(S) \geq g'(\psi(S))$ for all $S \subseteq H$. Similarly, $g'(S') \geq g(\psi'(S'))$ for all $S' \subseteq H'$. In particular, if $S \subseteq H$, then $g(S) \geq g'(\psi(S)) \geq g(\psi'\psi(S))$. By induction it follows that

$$g(S) \geq g'(\psi(S)) \geq g((\psi'\psi)^t(S))$$

for all $t \geq 1$. But $\psi'\psi : H \to H$ is a bijection, so it permutes the subsets of H. Thus there is an integer $t \geq 1$ such that $(\psi'\psi)^t(S) = S$ for all $S \subseteq H$, and $g(S) = g'(\psi(S))$ for all $S \subseteq H$.

We now use induction on $|H - S|$ to show that $e(S) = e'(\psi(S))$ for all $S \subseteq H$. If $|H - S| = 0$, then $S = H$ and $\psi(S) = H'$. But $e(H) = g(H) = g'(H') = e'(H')$, so the result holds in this case.

Now assume that $S \subseteq H$ and $|H - S| > 0$. Let $S_1, \dots, S_r$ be all of the distinct subsets of H that contain S properly. Then $\psi(S_1), \dots, \psi(S_r)$ are all of the distinct subsets of H' that contain $\psi(S)$ properly. By induction it follows that $e(S_i) = e'(\psi(S_i))$ for all i, so

$$e(S) = g(S) - \sum_{i=1}^{r} e(S_i) = g'(\psi(S)) - \sum_{i=1}^{r} e'(\psi(S_i)) = e'(\psi(S)).$$

Similarly, $e'(S') = e(\psi'(S'))$ for all $S' \subseteq H'$, and this completes the proof.

THEOREM 3.10. *Assume that R is a field. Let P and P' be finite posets, and let $f_\bullet : A_\bullet(P) \to A_\bullet(P')$ be an isomorphism. Then there exist isomorphisms of posets*

$\phi : P \to P'$ and $\phi' : P' \to P$ and tall orders $x_0, \dots, x_n$ on P_0 and $x'_0, \dots, x'_n$ on P'_0 such that $x_0, \dots, x_n$ is $f_\bullet$-compatible with $0, \phi(x_1), \dots, \phi(x_n)$ and $x'_0, \dots, x'_n$ is $f_\bullet^{-1}$-compatible with $0, \phi'(x'_1), \dots, \phi'(x'_n)$.

PROOF. The proof proceeds by induction on $h(P)$. First suppose that $h(P) = 1$. By Proposition 3.7 there are $f_\bullet$-compatible orderings $x_0, \dots, x_n$ of P_0 and $y'_0, \dots, y'_n$ of P'_0. Define $\phi : P \to P'$ by setting $\phi(x_i) = y'_i$ for $1 \le i \le n$. Then ϕ is an isomorphism of posets having the desired properties. The same argument applied to $f_\bullet^{-1}$ gives the isomorphism $\phi' : P' \to P$.

Now suppose that $h(P) > 1$. Let $H \subseteq P$ and $H' \subseteq P'$ be the subposets consisting of all elements that are not of maximum height. Then $h(H) = h(P)-1$, and $f_\bullet$ restricts to an isomorphism $h_\bullet : A_\bullet(H) \to A_\bullet(H')$. By induction there are isomorphisms of posets $\psi : H \to H'$ and $\psi' : H' \to H$ and tall orders $x_0, \dots, x_m$ on H_0 and $x'_0, \dots, x'_m$ on H'_0 such that $x_0, \dots, x_m$ is $h_\bullet$-compatible with $0, \psi(x_1), \dots, \psi(x_m)$ and $x'_0, \dots, x'_m$ is $h_\bullet^{-1}$-compatible with $0, \psi'(x'_1), \dots, \psi'(x'_m)$.

Write the power set $\mathscr{P}(H)$ of H as $\mathscr{P}(H) = \{S_1, \dots, S_{2^m}\}$, where the subsets $S_1, \dots, S_{2^m}$ are indexed so that $|S_1| \le \cdots \le |S_{2^m}|$. For $1 \le i \le 2^m$ set

$$T_i = \{y \in P - H \mid S_i = P_{<y}\} \quad \text{and} \quad T'_i = \{y' \in P' - H' \mid \psi(S_i) = P'_{<y'}\}.$$

Then $P - H$ is the disjoint union of $T_1, \dots, T_{2^m}$, and $P' - H'$ is the disjoint union of $T'_1, \dots, T'_{2^m}$. Moreover, $|T_i| = |T'_i|$ for all i by Lemma 3.9.

Choose an ordering $x_{m+1}, \dots, x_n$ on $P - H$ such that if $x_s \in T_i$, $x_t \in T_j$, and $i < j$, then $s < t$. Similarly, choose an ordering $y'_{m+1}, \dots, y'_n$ on $P' - H'$ such that if $y'_s \in T'_i$, $y'_t \in T'_j$, and $i < j$, then $s < t$. Let C denote the matrix of f_1, and assume that C is written with respect to the ordered bases $[x_0], \dots, [x_n]$ of $A_1(P)$ and $[0], [\psi(x_1)], \dots, [\psi(x_m)], [y'_{m+1}], \dots, [y'_n]$ of $A_1(P')$. Then C is a block upper triangular matrix: the first diagonal block C_1 has columns indexed by $[x_0], \dots, [x_m]$ and rows indexed by $[0], [\psi(x_1)], \dots, [\psi(x_m)]$; the other diagonal block C_2 has columns indexed by $[x_{m+1}], \dots, [x_n]$ and rows indexed by $[y'_{m+1}], \dots, [y'_n]$. In particular, $\det C = (\det C_1)(\det C_2)$.

Suppose that $y' \in T'_i$ and $y \in T_j$ are elements with $c_{y'y} \neq 0$. If $x \in S_i$, then $\psi(x) < y'$. Because $x_0, \dots, x_m$ is $h_\bullet$-compatible with $0, \psi(x_1), \dots, \psi(x_m)$, it follows that $\psi(x) \in \operatorname{supp} h_1[x] = \operatorname{supp} f_1[x]$ and hence $x < y$ by Lemma 3.8. Then $S_i < y$ so that $S_i \subseteq P_{<y} = S_j$. Hence $i \le j$, and the submatrix C_2 is itself block upper triangular: the i^{th} diagonal block of C_2 has columns indexed by elements in T_i and rows indexed by elements in T'_i.

Let $x \in P_0$ and $x' \in P'$. If $x \in H_0$, set $\tilde{c}_{x'x} = c_{x'x}$; if $x \in T_i$ and $x' \in T'_i$, set $\tilde{c}_{x'x} = c_{x'x}$; and if $x \in T_i$ and $x' \in P' - T'_i$, set $\tilde{c}_{x'x} = 0$. Finally, set

$$\tilde{c}_{0x} = 1 - \sum_{x' \in P'} \tilde{c}_{x'x}$$

for all $x \in P_0$. By Proposition 2.6 the matrix $\tilde{C} = (\tilde{c}_{x'x})$ determines a homomorphism $\tilde{f}_\bullet : A_\bullet(P) \to A_\bullet(P')$. Because $\tilde{C}$ is a block upper triangular matrix with the same diagonal blocks as C, it follows that $\det \tilde{C} = \det C \neq 0$. Thus $\tilde{f}_\bullet$ is an isomorphism. Moreover, $\tilde{f}_\bullet$ restricts to an isomorphism $\tilde{f}_\bullet : A_\bullet(T_i) \to A_\bullet(T_i')$ for all i. Let $0 = t_{i0}, t_{i1}, \dots, t_{im_i}$ be the ordering on $(T_i)_0$ obtained by regarding T_i as a subset of the ordered set $P - H = \{x_{m+1}, \dots, x_n\}$. By Proposition 3.7 there is an $\tilde{f}_\bullet$-compatible ordering $t'_{i0}, \dots, t'_{im_i}$ of $(T_i')_0$. Then the function $\psi_i : T_i \to T_i'$ given by $\psi_i(t_{ij}) = t'_{ij}$ for $1 \leq j \leq m_i$ is a bijection.

Because $P - H$ is the disjoint union of $T_1, \dots, T_{2^m}$, it is possible to define a function $\phi : P \to P'$ by setting

$$\phi(x) = \begin{cases} \psi(x) & \text{if } x \in H \\ \psi_i(x) & \text{if } x \in T_i, \end{cases}$$

and it is clear that ϕ is a bijection. Suppose that $x < y$ in P. If $x, y \in H$, then $\phi(x) < \phi(y)$ because ψ is an isomorphism of posets. If x and y are not both in H, then $x \in S_i$ and $y \in T_i$ for some i. Then $\phi(y) = \psi_i(y) \in T_i'$, so $\psi(S_i) < \phi(y)$. But $\phi(x) = \psi(x) \in \psi(S_i)$, so $\phi(x) < \phi(y)$. Hence ϕ is an isomorphism of posets.

Finally, the ordering $x_0, \dots, x_m$ of H_0 is $h_\bullet$-compatible with $0, \phi(x_1), \dots, \phi(x_m)$, and for each i the orderings $t_{i0}, \dots, t_{im_i}$ of $(T_i)_0$ and $0, \phi(t_{i1}), \dots, \phi(t_{im_i})$ of $(T_i')_0$ are $\tilde{f}_\bullet$-compatible. It follows that the ordering $x_0, \dots, x_n$ of P_0 is $f_\bullet$-compatible with the ordering $0, \phi(x_1), \dots, \phi(x_n)$ of P_0'.

The same argument shows that there exist an isomorphism of posets $\phi' : P' \to P$ and a tall order $x_0', \dots, x_n'$ on P_0' that is $f_\bullet^{-1}$-compatible with the ordering $0, \phi'(x_1'), \dots, \phi'(x_n')$, and this completes the proof.

COROLLARY 3.11. *If P and P' are finite partially ordered sets such that $A_\bullet(P) \cong A_\bullet(P')$, then $P \cong P'$.*

PROOF. By working over the quotient field of R, we may assume that R is itself a field. Then the result follows immediately from Theorem 3.10.

4. Annihilators and the graded center

The purpose of this section is to give a description of the graded center of $A_\bullet(P)$ in terms of the elements that annihilate all homogeneous elements of positive degree in $A_\bullet(P)$. Recall that the graded center $Z_\bullet(P)$ is defined to be the R-submodule generated by all homogeneous elements $z \in A_\bullet(P)$ such that $az = (-1)^{(\deg a)(\deg z)} za$

for all homogeneous elements $a \in A_\bullet(P)$. Note that if $z \in Z_m(P)$ and $a \in A_n(P)$ are any two homogeneous elements, then

$$\begin{aligned}(da)z + (-1)^n a(dz) &= d(az)\\ &= (-1)^{mn} d(za)\\ &= (-1)^{mn}(dz)a + (-1)^{m(n-1)} z(da)\\ &= (-1)^{mn}(dz)a + (da)z.\end{aligned}$$

Hence $a(dz) = (-1)^{(m-1)n}(dz)a$, and it follows that $dz \in Z_{m-1}(P)$. Thus $Z_\bullet(P)$ is a differential graded $A_\bullet(\emptyset)$-subalgebra of $A_\bullet(P)$.

If S is any subset of $A_\bullet(P)$, then Ann S will denote the ideal consisting of all two-sided annihilators of S; in other words,

$$\text{Ann } S = \{x \in A_\bullet(P) \mid xs = sx = 0 \text{ for all } s \in S\}.$$

Let $A_+(P)$ denote the ideal of $A_\bullet(P)$ generated by all homogeneous elements of positive degree. Then the annihilator Ann $A_+(P) =$ Ann $A_1(P)$ is a homogeneous ideal of $A_\bullet(P)$. Let $I_\bullet(P)$ denote the differential graded ideal generated by Ann $A_+(P)$. The first result of this section gives an explicit description of Ann $A_+(P)$.

PROPOSITION 4.1. *Let P be a finite non-empty poset. Then* Ann $A_+(P)$ *is the span of all elements of the form* $[0 < m < \cdots < M]$, *where m is minimal and M is maximal in P. In particular, if P contains no connected components of height one, then $I_1(P) = 0$.*

PROOF. If m is minimal and M is maximal in P, then the definition of the multiplication in $A_\bullet(P)$ shows that $[0 < m < \cdots < M] \in$ Ann $A_+(P)$. Conversely, suppose that $x = \sum_{i=1}^{s} c_i[x_{0i} < \cdots < x_{ni}]$ is a homogeneous element of Ann $A_+(P)$ with $c_i \neq 0$ for $1 \leq i \leq s$. Because $[0]x = 0$, it follows that $x_{0i} = 0$ for all i. If $n = 0$, then it is easy to see that P is empty, so we may assume that $n > 0$. Let m be a minimal element of P. Then

$$0 = [m]x = -\sum_{i=1}^{s} c_i[0 < m < x_{1i} < \cdots < x_{ni}],$$

and it follows that $m \not< x_{1i}$ for all i. Because this relation holds for every minimal element m of P, we conclude that x_{1i} is minimal for all i. Similarly, if M is a maximal element of P, then the fact that $0 = x[M]$ implies that x_{ni} is maximal for all i. This proves the first statement, and the second follows easily.

PROPOSITION 4.2. *Let P be a finite non-empty poset. If a and b are homogeneous elements of $I_\bullet(P)$, then $ab = 0$.*

PROOF. Because $a, b \in I_\bullet(P)$, it is possible to write $a = a' + da''$ and $b = b' + db''$ for some homogeneous elements $a', a'', b', b'' \in \operatorname{Ann} A_+(P) \subseteq A_+(P)$. Then

$$ab = (a' + da'')(b' + db'') = (da'')(db'') = d\big(a''(db'')\big) = 0,$$

as desired.

PROPOSITION 4.3. *Let P be a finite poset. Then $Z_\bullet(P)$ is the differential graded $A_\bullet(\emptyset)$-algebra generated by* Ann $A_+(P)$. *Moreover, if P is non-empty, then $Z_\bullet(P) = A_\bullet(\emptyset) \oplus I_\bullet(P)$ as graded R-modules.*

PROOF. We begin by showing that $Z_\bullet(P) = A_\bullet(\emptyset) + I_\bullet(P)$. It is clear that $A_\bullet(\emptyset) + I_\bullet(P) \subseteq Z_\bullet(P)$, and we will prove that $Z_n(P) = A_n(\emptyset) + I_n(P)$ for all n by downward induction on n. If N is the largest degree such that $A_N(P) \neq 0$, then certainly $Z_n(P) = A_n(\emptyset) + I_n(P) = 0$ for all $n > N$, and $Z_N(P) = A_N(P) = A_N(\emptyset) + I_N(P)$.

Now suppose that $1 \leq n < N$ and that $Z_{n+1}(P) = A_{n+1}(\emptyset) + I_{n+1}(P)$. Let $x \in Z_n(P)$. Then $x = [0](dx) + d\big([0]x\big)$, and by induction $[0]x \in Z_{n+1}(P) = A_{n+1}(\emptyset) + I_{n+1}(P) = I_{n+1}(P)$. Hence $d\big([0]x\big) \in I_n(P)$, and it suffices to show that $[0](dx) \in A_n(\emptyset) + I_n(P)$. If $n = 1$, then $[0](dx)$ is a multiple of $[0]$, so it lies in $A_1(\emptyset)$. Thus we may assume that $2 \leq n < N$. Write $dx = \sum_{i=1}^{s} c_i[x_{1i} < \cdots < x_{n-1,i}]$, and let $y \in P_0$. Then

$$\begin{aligned}\sum_{i=1}^{s}(-1)^{n-1}c_i[0 < x_{1i} < \cdots < x_{n-1,i} < y] &= \sum_{i=1}^{s} c_i[x_{1i} < \cdots < x_{n-1,i}][0 < y] \\ &= (dx)[0][y] = [0][y](dx) \\ &= \sum_{i=1}^{s} c_i[0 < y][x_{1i} < \cdots < x_{n-1,i}].\end{aligned}$$

If any term in this last sum is non-zero, then it follows that $c_j[0 < y < x_{1j} < \cdots < x_{n-1,j}] \neq 0$ for some j with $1 \leq j \leq s$. But such a term cannot occur in the sum $\sum_i (-1)^{n-1}c_i[0 < x_{1i} < \cdots < x_{n-1,i} < y]$ because $n \geq 2$. Thus $[y][0](dx) = (-1)^n[0](dx)[y] = -[0][y](dx) = 0$, and it follows that $[0](dx) \in A_n(P) \cap \operatorname{Ann} A_1(P) \subseteq I_n(P)$. Hence $Z_n(P) = A_n(\emptyset) + I_n(P)$ for all $n \geq 1$. But $Z_0(P) = A_0(P) = A_0(\emptyset) + I_0(P)$, so $Z_\bullet(P) = A_\bullet(\emptyset) + I_\bullet(P)$, as desired.

To show that the sum $A_\bullet(\emptyset) + I_\bullet(P)$ is direct when P is non-empty, it suffices to show that $I_0(P) = 0$ and $R[0] \cap I_1(P) = 0$. Both of these facts follow easily from Proposition 4.1.

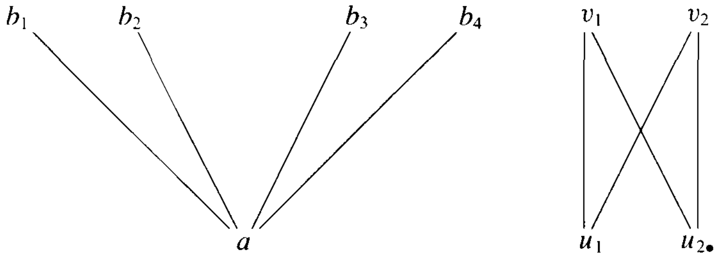
If P is a finite non-empty poset, let P^* denote the dual of P. By Proposition 4.1 there is an R-linear map $f_\bullet : \operatorname{Ann} A_+(P) \to \operatorname{Ann} A_+(P^*)$ satisfying

$$f_\bullet[0 < m < \cdots < M] = [0 < M < \cdots < m],$$

and $f_\bullet$ extends uniquely to an isomorphism of differential graded $A_\bullet(\emptyset)$-algebras $f_\bullet : Z_\bullet(P) \to Z_\bullet(P^*)$ by Proposition 4.3. Thus we obtain the following result.

COROLLARY 4.4. *If P is a finite poset, then $Z_\bullet(P) \cong Z_\bullet(P^*)$.*

It often happens, however, that two posets P and Q satisfy $Z_\bullet(P) \cong Z_\bullet(Q)$ even when $Q \ncong P$ and $Q \ncong P^*$. Such an example is given by the following posets P and Q:



Indeed, Ann $A_+(P)$ is given by the span of $\{[0 < a < b_i] \mid 1 \le i \le 4\}$, whereas Ann $A_+(Q)$ is given by the span of $\{[0 < u_i < v_j] \mid 1 \le i, j \le 2\}$. If f is any bijection between these sets, then it is easy to see that f extends uniquely to a differential graded $A_\bullet(\emptyset)$-isomorphism between $Z_\bullet(P)$ and $Z_\bullet(Q)$.

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