Factors from Buildings

Jacqui Ramagge and Guyan Robertson

Abstract. We discuss some group actions on boundaries of thick buildings of order q. These actions are hyperfinite of type III_λ where $\lambda = q^{-1}$ if Γ is a free group acting on a tree and $\lambda = q^{-2}$ if Γ is a triangle group acting on an $\tilde{A}_2$ building. Detailed proofs of these results can be found in [**RR1**] and [**RR2**].

1. The Buildings

The only affine buildings which admit a simply transitive group action on the vertices are those of the form $\tilde{A}_{n_1} \times \cdots \times \tilde{A}_{n_k}$. We consider $\tilde{A}_1$ buildings, or trees, and $\tilde{A}_2$ buildings, or triangle buildings. We conjecture that similar results hold for $\tilde{A}_n$ buildings with $\lambda = q^{-(n \mod 2)}$.

An $\tilde{A}_n$ building Δ is a simplicial complex of rank 2; its maximal simplices are called **chambers**. We assume that each edge lies on precisely $q+1$ chambers for some integer $q \geq 2$ called the **order** of Δ. This assumption is no stronger than assuming each edge lies on a finite number of chambers if $n \geq 2$. An **apartment** $\mathcal{A}$ in Δ is a subcomplex isomorphic to the Coxeter complex of the same type. If $n = 1$ this is a line tesselated by unit intervals; if $n = 2$ an apartment is a plane tesselated by equilateral triangles. A **sector** in Δ is a simplicial cone in some apartment $\mathcal{A} \subseteq \Delta$. If $n = 1$ simplicial cones are semi-infinite paths. Two sectors are **equivalent** if they contain a common subsector. This corresponds to tail equivalence of semi-infinite paths if $n = 1$. Let Ω be the set of equivalence classes of sectors in Δ. Then Ω is the set of chambers of the spherical building at infinity associated to Δ, called the **boundary** of Δ. Given any vertex $v \in \Delta$ and any equivalence class ω of sectors there is a unique sector $\mathcal{S}_v(\omega)$ in the equivalence class ω based at v. Thus Ω can be identified with the set of sectors emanating from a given vertex $v \in \Delta$. There is a topology on Ω which in the case $n = 1$ reduces to the standard topology on the boundary of a tree. A basis for this topology consists of sets of equivalence classes of sectors whose representatives based at a fixed vertex $v \in \Delta$ share a common initial segment. With respect to this topology, Ω is a totally disconnected compact Hausdorff space. For each vertex $v \in \Delta$ there is a natural Borel probability measure ν_v on Ω [**CMS**]. Given any two vertices $u, v \in \Delta$ the measures ν_u and ν_v are mutually absolutely continuous. We refer to [**B, C, R, S**] for more details on buildings.

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2. The Groups and the Actions

Given an $\tilde{A}_n$ building Δ, there is a type τ defined on the vertices of Δ such that $\tau(v) \in \mathbb{Z}/n\mathbb{Z}$ for each vertex $v \in \Delta$. An automorphism g of Δ is **type-rotating** if there exists an $i \in \{0, 1, \dots, n-1\}$ such that $\tau(gv) = \tau(v) + i$ for all vertices $v \in \Delta$. Denote by $\mathrm{Aut}_{\mathrm{tr}}(\Delta)$ the group of all type-rotating automorphisms of Δ. A group Γ is an $\tilde{A}_n \times \cdots \times \tilde{A}_n$ **group** if $\Gamma \le \mathrm{Aut}_{\mathrm{tr}}(\Delta)$ for some thick $\tilde{A}_n \times \cdots \times \tilde{A}_n$ building. Such groups have simple presentations.

By [**T**], any $\tilde{A}_1$ group is of the form $\Gamma \cong \underbrace{\mathbb{Z}_2 * \cdots * \mathbb{Z}_2}_{s \text{ terms}} * \underbrace{\mathbb{Z} * \cdots * \mathbb{Z}}_{t \text{ terms}}$ where $s + 2t = q + 1$, so that

$$\Gamma = \langle a_1, \dots, a_{s+t} \;:\; a_i^2 = 1 \text{ for } i \in \{1, \dots, s\} \rangle.$$

By [**CMSZ**], any $\tilde{A}_2$ group Γ has a presentation of the form

$$\Gamma = \langle a_i \;;\; a_x a_y a_z = 1 \text{ for } (x, y, z) \in \mathcal{T} \rangle$$

where $\mathcal{T}$ is a triella. The Cayley graph of these groups constructed by right multiplication with respect to these generators can be identified with the one-skeleton of the corresponding building Δ. Under this identification, the action of Γ on Δ corresponds to the action of Γ on its Cayley graph by left multiplication.

The action of Γ on Δ induces an action of Γ on Ω. Under this action the measure ν_v is quasi-invariant for each vertex $v \in \Delta$ so that we obtain an action of Γ on $L^\infty(\Omega, \nu_v)$. We prove that the action of Γ on Ω is free, ergodic, amenable, and of type III_λ with $\lambda = q^{-i}$ where $i = n \mod 2$. In particular, this proves that the von Neumann algebra $L^\infty(\Omega, \nu_v) \rtimes \Gamma$ is the hyperfinite factor of type III_λ. See [**Su**] for details on von Neumann algebras.

3. Proof Outlines

In the case $n = 1$ the freeness of the action of Γ on Ω follows from the fact that the only boundary points fixed under left multiplication by $g \in \Gamma \setminus \{e\}$ correspond to the semi-infinite words $\omega = ggg\cdots$ and $\omega' = g^{-1}g^{-1}g^{-1}\cdots$. In particular these words are periodic. We broaden the notions of periodic apartments introduced in [**M, MZ**] and prove that a boundary point fixed by any $g \in \Gamma \setminus \{e\}$ must have a representative in a periodic apartment. We denote by Π the set of all such **periodic limit points**. We prove that Π is a set of measure zero in Ω and deduce that the action of Γ on Ω is free.

We prove ergodicity of the action by showing that any Γ-invariant function $f \in L^\infty(\Omega, \nu_v)$ is constant a.e. The proof of this result for the case $n = 1$ can be found in [**PS**] and our proof in the case $n = 2$ is a generalization of theirs. We use a condition in [**A**] to establish the amenability if the action of Γ on Ω, and we show that it is of type III_λ by a detailed analysis of the **Krieger ratio set** $r(\Gamma)$ of essential values of the Radon-Nikodym derivatives $\dfrac{d\nu_v}{d\nu_{gv}}$ (see [**Su**]).

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Mathematics Dept, University of Newcastle, NSW 2308, Australia
E-mail address: jacqui@maths.newcastle.edu.au
E-mail address: guyan@maths.newcastle.edu.au