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## FACTORS FROM TREES

JACQUI RAMAGGE AND GUYAN ROBERTSON

(Communicated by Palle E. T. Jorgensen)

**ABSTRACT.** We construct factors of type  $\text{III}_{1/n}$  for  $n \in \mathbb{N}, n \geq 2$ , from group actions on homogeneous trees and their boundaries. Our result is a discrete analogue of a result of R.J Spatzier, where the hyperfinite factor of type  $\text{III}_1$  is constructed from a group action on the boundary of the universal cover of a manifold.

### 1. INTRODUCTION

Let  $\Gamma$  be a group acting simply transitively on the vertices of a homogeneous tree  $\mathcal{T}$  of degree  $n + 1 < \infty$ . Then, by [FTN, Ch. I, Theorem 6.3],

$$\Gamma \cong \mathbb{Z}_2 * \cdots * \mathbb{Z}_2 * \mathbb{Z} * \cdots * \mathbb{Z}$$

where there are  $s$  factors of  $\mathbb{Z}_2$ ,  $t$  factors of  $\mathbb{Z}$ , and  $s + 2t = n + 1$ . Thus  $\Gamma$  has a presentation

$$\Gamma = \langle a_1, \dots, a_{s+t} : a_i^2 = 1 \text{ for } i \in \{1, \dots, s\} \rangle,$$

we can identify the Cayley graph of  $\Gamma$  constructed via right multiplication with  $\mathcal{T}$  and the action of  $\Gamma$  on  $\mathcal{T}$  is equivalent to the natural action of  $\Gamma$  on its Cayley graph via left multiplication.

We can associate a natural boundary to  $\mathcal{T}$ , namely the set  $\Omega$  of semi-infinite reduced words in the generators of  $\Gamma$ . The action of  $\Gamma$  on  $\mathcal{T}$  induces an action of  $\Gamma$  on  $\Omega$ .

For each  $x \in \Gamma$ , let

$$\Omega^x = \{ \omega \in \Omega : \omega = x \cdots \}$$

be the set of semi-infinite reduced words beginning with  $x$ . The set  $\{ \Omega^x \}_{x \in \Gamma}$  is a set of basic open sets for a compact Hausdorff topology on  $\Omega$ . Denote by  $|x|$  the length of a reduced expression for  $x$ . Let  $V^m = \{ x \in \Gamma : |x| = m \}$  and define  $N_m = |V^m|$ . Then  $\Omega$  is the disjoint union of the  $N_m$  sets  $\Omega^x$  for  $x \in V^m$ .

We can also endow  $\Omega$  with the structure of a measure space.  $\Omega$  has a unique distinguished Borel probability measure  $\nu$  such that

$$\nu(\Omega^x) = \frac{1}{n+1} \left( \frac{1}{n} \right)^{|x|-1}$$

for every nontrivial  $x \in \Gamma$ . The sets  $\Omega^x$ ,  $x \in \Gamma$ , generate the Borel  $\sigma$ -algebra.

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This measure  $\nu$  on  $\Omega$  is quasi-invariant under the action of  $\Gamma$ , so that  $\Gamma$  acts on the measure space  $(\Omega, \nu)$  and enables us to extend the action of  $\Gamma$  to an action on  $L^\infty(\Omega, \nu)$  via

$$g \cdot f(\omega) = f(g^{-1} \cdot \omega)$$

for all  $g \in \Gamma$ ,  $f \in L^\infty(\Omega, \nu)$ , and  $\omega \in \Omega$ . We may therefore consider the von Neumann algebra  $L^\infty(\Omega, \nu) \rtimes \Gamma$  which we shall write as  $L^\infty(\Omega) \rtimes \Gamma$  for brevity.

2. THE FACTORS

We note that the action of  $\Gamma$  on  $\Omega$  is free since if  $g\omega = \omega$  for some  $g \in \Gamma$  and  $\omega \in \Omega$  then we must have either  $\omega = ggg \dots$  or  $\omega = g^{-1}g^{-1}g^{-1} \dots$  and

$$\nu \{ ggg \dots, g^{-1}g^{-1}g^{-1} \dots \} = 0.$$

The action of  $\Gamma$  on  $\Omega$  is also ergodic by the proof of [PS, Proposition 3.9], so that  $L^\infty(\Omega) \rtimes \Gamma$  is a factor. Establishing the type of the factor is not quite as straightforward. We begin by recalling some classical definitions.

**Definition 2.1.** Given a group  $\Gamma$  acting on a measure space  $\Omega$ , we define the **full group**,  $[\Gamma]$ , of  $\Gamma$  by

$$[\Gamma] = \{ T \in \text{Aut}(\Omega) : T\omega \in \Gamma\omega \text{ for almost every } \omega \in \Omega \}.$$

The set  $[\Gamma]_0$  of measure preserving maps in  $[\Gamma]$  is then given by

$$[\Gamma]_0 = \{ T \in [\Gamma] : T \circ \nu = \nu \}.$$

**Definition 2.2.** Let  $G$  be a countable group of automorphisms of the measure space  $(\Omega, \nu)$ . Following W. Krieger, define the **ratio set**  $r(G)$  to be the subset of  $[0, \infty)$  such that if  $\lambda \geq 0$  then  $\lambda \in r(G)$  if and only if for every  $\epsilon > 0$  and Borel set  $\mathcal{E}$  with  $\nu(\mathcal{E}) > 0$ , there exist a  $g \in G$  and a Borel set  $\mathcal{F}$  such that  $\nu(\mathcal{F}) > 0$ ,  $\mathcal{F} \cup g\mathcal{F} \subseteq \mathcal{E}$  and

$$\left| \frac{d\nu \circ g}{d\nu}(\omega) - \lambda \right| < \epsilon$$

for all  $\omega \in \mathcal{F}$ .

*Remark 2.3.* The ratio set  $r(G)$  depends only on the quasi-equivalence class of the measure  $\nu$ ; see [HO, §I-3, Lemma 14]. It also depends only on the full group in the sense that

$$[H] = [G] \Rightarrow r(H) = r(G).$$

The following result will be applied in the special case where  $G = \Gamma$ . However, since the simple transitivity of the action doesn't play a role in the proof, we can state it in greater generality.

**Proposition 2.4.** *Let  $G$  be a countable subgroup of  $\text{Aut}(T) \leq \text{Aut}(\Omega)$ . Suppose there exist an element  $g \in G$  such that  $d(ge, e) = 1$  and a subgroup  $K$  of  $[G]_0$  whose action on  $\Omega$  is ergodic. Then*

$$r(G) = \{ n^k : k \in \mathbb{Z} \} \cup \{ 0 \}.$$

*Proof.* By Remark 2.3, it is sufficient to prove the statement for some group  $H$  such that  $[H] = [G]$ . In particular, since  $[G] = [\langle G, K \rangle]$  for any subgroup  $K$  of  $[G]_0$ , we may assume without loss of generality that  $K \leq G$ .

By [FTN, Chapter II, part 1]), for each  $g \in G$  and  $\omega \in \Omega$  we have

$$\frac{d\nu \circ g}{d\nu}(\omega) \in \{ n^k : k \in \mathbb{Z} \} \cup \{ 0 \}.$$

Since  $G$  acts ergodically on  $\Omega$ ,  $r(G) \setminus \{0\}$  is a group. It is therefore enough to show that  $n \in r(G)$ . Write  $x = ge$  and note that  $\nu_x = \nu \circ g^{-1}$ . By [FTN, Chapter II, part 1)] we have

$$(1) \quad \frac{d\nu_x}{d\nu}(\omega) = n, \text{ for all } \omega \in \Omega_e^x.$$

Let  $\mathcal{E} \subseteq \Omega$  be a Borel set with  $\nu(\mathcal{E}) > 0$ . By the ergodicity of  $K$ , there exist  $k_1, k_2 \in K$  such that the set

$$\mathcal{F} = \{\omega \in \mathcal{E} : k_1\omega \in \Omega_e^x \text{ and } k_2g^{-1}k_1\omega \in \mathcal{E}\}$$

has positive measure.

Finally, let  $t = k_2g^{-1}k_1 \in G$ . By construction,  $\mathcal{F} \cup t\mathcal{F} \subseteq \mathcal{E}$ . Moreover, since  $K$  is measure-preserving,

$$\frac{d\nu \circ t}{d\nu}(\omega) = \frac{d\nu \circ g^{-1}}{d\nu}(k_1\omega) = \frac{d\nu_x}{d\nu}(k_1\omega) = n \text{ for all } \omega \in \mathcal{F}$$

by (1), since  $k_1 \in \Omega_e^x$ . This proves  $n \in r(G)$ , as required.  $\square$

**Corollary 2.5.** *If, in addition to the hypotheses for Proposition 2.4, the action of  $G$  is free, then  $L^\infty(\Omega) \rtimes G$  is a factor of type III $_{1/n}$ .*

*Proof.* Having determined the ratio set, this is immediate from [C1, Corollaire 3.3.4].  $\square$

Thus, if we can find a countable subgroup  $K \leq [\Gamma]_0$  whose action on  $\Omega$  is ergodic we will have shown that  $L^\infty(\Omega) \rtimes \Gamma$  is a factor of type III $_{1/n}$ . To this end, we prove the following sufficiency condition for ergodicity.

**Lemma 2.6.** *Let  $K$  be group which acts on  $\Omega$ . If  $K$  acts transitively on the collection of sets  $\{\Omega^x : x \in \Gamma, |x| = m\}$  for each natural number  $m$ , then  $K$  acts ergodically on  $\Omega$ .*

*Proof.* Suppose that  $X_0 \subseteq \Omega$  is a Borel set which is invariant under  $K$  and such that  $\nu(X_0) > 0$ . We show that this necessarily implies  $\nu(\Omega \setminus X_0) = 0$ , thus establishing the ergodicity of the action.

Define a new measure  $\mu$  on  $\Omega$  by  $\mu(X) = \nu(X \cap X_0)$  for each Borel set  $X \subseteq \Omega$ . Now, for each  $g \in K$ ,

$$\begin{aligned} \mu(gX) &= \nu(gX \cap X_0) = \nu(X \cap g^{-1}X_0) \\ &= \nu(X \cap X_0) \\ &= \mu(X), \end{aligned}$$

and therefore  $\mu$  is  $K$ -invariant. Since  $K$  acts transitively on the basic open sets  $\Omega^x$  associated to words  $x$  of length  $m$  this implies that

$$\mu(\Omega^x) = \mu(\Omega^y)$$

whenever  $|x| = |y|$ . Since  $\Omega$  is the union of  $N_m$  disjoint sets  $\Omega^x$ ,  $x \in V^m$ , each of which has equal measure with respect to  $\mu$ , we deduce that

$$\mu(\Omega^x) = \frac{c}{N_m}$$

for each  $x \in V^m$ , where  $c = \mu(X_0) = \nu(X_0) > 0$ . Thus  $\mu(\Omega^x) = c\nu(\Omega^x)$  for every  $x \in \Gamma$ .

Since the sets  $\Omega^x$ ,  $x \in \Gamma$  generate the Borel  $\sigma$ -algebra, we deduce that  $\mu(X) = c\nu(X)$  for each Borel set  $X$ . Therefore

$$\begin{aligned} \nu(\Omega \setminus X_0) &= c^{-1}\mu(\Omega \setminus X_0) \\ &= c^{-1}\nu((\Omega \setminus X_0) \cap X_0) = 0, \end{aligned}$$

thus proving ergodicity. □

In the last of our technical results, we give a constructive proof of the existence of a countable ergodic subgroup of  $[\Gamma]_0$ .

**Lemma 2.7.** *There is a countable ergodic group  $K \leq \text{Aut}(\Omega)$  such that  $K \leq [\Gamma]_0$ .*

*Proof.* Let  $x, y \in V^m$ . We construct a measure preserving automorphism  $k_{x,y}$  of  $\Omega$  such that

- (1)  $k_{x,y}$  is almost everywhere a bijection from  $\Omega^x$  onto  $\Omega^y$ ,
- (2)  $k_{x,y}$  is the identity on  $\Omega \setminus (\Omega^x \cup \Omega^y)$ .

It then follows from Lemma 2.6 that the group

$$K = \langle k_{x,y} : \{x, y\} \subseteq V^m, m \in \mathbb{N} \rangle$$

acts ergodically on  $\Omega$  and the construction will show explicitly that  $K \leq [\Gamma]_0$ .

Fix  $x, y \in V^m$  and suppose that we have reduced expressions  $x = x_1 \dots x_m$ , and  $y = y_1 \dots y_m$ .

Define  $k_{x,y}$  to be left multiplication by  $yx^{-1}$  on each of the sets  $\Omega^{xz}$  where  $|z| = 1$  and  $z \notin \{x_m^{-1}, y_m^{-1}\}$ . Then  $k_{x,y}$  is a measure preserving bijection from each such set onto  $\Omega^{yz}$ . If  $y_m = x_m$  then  $k_{x,y}$  is now well defined everywhere on  $\Omega^x$ .

Suppose now that  $y_m \neq x_m$ . Then  $k_{x,y}$  is defined on the set  $\Omega^x \setminus \Omega^{xy_m^{-1}}$ , which it maps bijectively onto  $\Omega^y \setminus \Omega^{yx_m^{-1}}$ . Now define  $k_{x,y}$  to be left multiplication by  $yx_m^{-1}y_mx^{-1}$  on each of the sets  $\Omega^{xy_m^{-1}z}$  where  $|z| = 1$  and  $z \notin \{x_m, y_m\}$ . Then  $k_{x,y}$  is a measure preserving bijection of each such  $\Omega^{xy_m^{-1}z}$  onto  $\Omega^{yx_m^{-1}z}$ .

Thus we have extended the domain of  $k_{x,y}$  so that it is now defined on the set  $\Omega^x \setminus \Omega^{xy_m^{-1}x_m}$ , which it maps bijectively onto  $\Omega^y \setminus \Omega^{yx_m^{-1}y_m}$ .

Next define  $k_{x,y}$  to be left multiplication by  $yx_m^{-1}y_mx_m^{-1}y_mx^{-1}$  on the sets  $\Omega^{xy_m^{-1}x_mz}$  where  $|z| = 1$  and  $z \notin \{x_m^{-1}, y_m^{-1}\}$ .

Continue in this way. At the  $j$ th step  $k_{x,y}$  is a measure preserving bijection from  $\Omega^x \setminus X_j$  onto  $\Omega^y \setminus Y_j$  where  $\nu(X_j) \rightarrow 0$  as  $j \rightarrow \infty$  so that eventually  $k_{x,y}$  is defined almost everywhere on  $\Omega$ . Finally, define

$$k_{x,y}(xy_m^{-1}x_my_m^{-1}x_my_m^{-1}x_m \dots) = yx_m^{-1}y_mx_m^{-1}y_mx_m^{-1}y_m \dots$$

thus defining  $k_{x,y}$  everywhere on  $\Omega$  in such a way that its action is pointwise approximable by  $\Gamma$  almost everywhere. Hence

$$K = \langle k_{x,y} : \{x, y\} \subseteq V^m, m \in \mathbb{N} \rangle$$

is a countable group with an ergodic measure-preserving action on  $\Omega$  and  $K \leq [\Gamma]_0$ . □

We are now in a position to prove our main result.

**Theorem 2.8.** *The von Neumann algebra  $L^\infty(\Omega) \rtimes \Gamma$  is the hyperfinite factor of type III $_{1/n}$ .*

*Proof.* By applying Corollary 2.5 with  $G = \Gamma$ ,  $g \in \Gamma$  any generator of  $\Gamma$ , and  $K$  as in Lemma 2.7 we conclude that  $L^\infty(\Omega) \rtimes \Gamma$  is a factor of type  $\text{III}_{1/n}$ .

To see that the factor is hyperfinite simply note that the action of  $\Gamma$  is amenable as a result of [A, Theorem 5.1]. We refer to [C2, Theorem 4.4.1] for the uniqueness of the hyperfinite factor of type  $\text{III}_{1/n}$ .  $\square$

*Remark 2.9.* In [Sp1], Spielberg constructs  $\text{III}_\lambda$  factor states on the algebra  $\mathcal{O}_2$ . The reduced  $C^*$ -algebra  $C(\Omega) \rtimes_r \Gamma$  is a Cuntz-Krieger algebra  $\mathcal{O}_A$  by [Sp2]. What we have done is construct a type  $\text{III}_{1/n}$  factor state on some of these algebras  $\mathcal{O}_A$ .

*Remark 2.10.* From [C2, p. 476], we know that if  $\Gamma = \mathbb{Q} \rtimes \mathbb{Q}^*$  acts naturally on  $\mathbb{Q}_p$ , then the crossed product  $L^\infty(\mathbb{Q}_p) \rtimes \Gamma$  is the hyperfinite factor of type  $\text{III}_{1/p}$ . This may be proved geometrically as above by regarding the the boundary of the homogeneous tree of degree  $p + 1$  as the one point compactification of  $\mathbb{Q}_p$  as in [CKW].

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