

# Triangle Buildings and Actions of Type III<sub>1/q<sup>2</sup></sub>\*

Jacqui Ramagge<sup>†</sup> and Guyan Robertson<sup>‡</sup>

*Mathematics Department, University of Newcastle,  
Callaghan, New South Wales 2308, Australia*

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We study certain group actions on triangle buildings and their boundaries and some von Neumann algebras which can be constructed from them. In particular, for buildings of order  $q \geq 3$  certain natural actions on the boundary are hyperfinite of type III<sub>1/q<sup>2</sup></sub>. © 1996 Academic Press, Inc.

## 1. INTRODUCTION

We begin with a triangle building  $\mathcal{A}$  whose one-skeleton is the Cayley graph of a group  $\Gamma$ , so that  $\Gamma$  acts simply-transitively on the vertices of  $\mathcal{A}$  by left multiplication. The group  $\Gamma$  (often referred to as an  $\tilde{A}_2$  group) has a relatively simple combinatorial structure, yet also has Kazhdan's property (T) [CMS]. It is interesting to note that not all such groups  $\Gamma$  can be embedded naturally as lattices in  $\mathrm{PGL}(3, \mathbb{K})$  where  $\mathbb{K}$  is a local field [CMSZ, II, Sect. 8]. As a tool in our studies we introduce the notion of a periodic apartment in  $\mathcal{A}$ .

We prove several useful facts about periodic apartments and the periodic limit points they define on the boundary,  $\Omega$ , of  $\mathcal{A}$ . The boundary  $\Omega$  is a totally disconnected compact Hausdorff space and there is a family of mutually absolutely continuous Borel probability measures  $\{\nu_v\}$  on  $\Omega$  indexed by the vertices of  $\mathcal{A}$ . We show that periodic boundary points form a dense subset of the boundary of trivial measure with respect to this class. We also prove that the only boundary points stabilized by the action of  $\Gamma$  are the periodic limit points. This enables us to deduce that the action of  $\Gamma$  on  $\Omega$  is free. By analogy with the tree case we show that the action of  $\Gamma$  on  $\Omega$  is also ergodic, thus establishing that  $L^\infty(\Omega) \rtimes \Gamma$  is a factor. Since the action of  $\Gamma$  is amenable,  $L^\infty(\Omega) \rtimes \Gamma$  is a hyperfinite factor. In Theorem 4.10 we prove that, if  $q \geq 3$ ,  $L^\infty(\Omega) \rtimes \Gamma$  is in fact the hyperfinite

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<sup>†</sup> E-mail: jacqui@maths.newcastle.edu.au.

<sup>‡</sup> E-mail: guyan@maths.newcastle.edu.au.

factor of type III<sub>1/q<sup>2</sup></sub>. The corresponding result for homogeneous trees was proved in [RR].

In Section 4.2, we consider the action of some other groups  $G$  acting on triangle buildings and we show that, under certain conditions, the action is of type III<sub>1/q<sup>2</sup></sub>. In particular, Theorem 4.13 asserts that if  $\Omega$  is the boundary of the building associated to  $\text{PGL}(3, \mathbb{F}_q((X)))$ , then the action of  $\text{PGL}(3, \mathbb{F}_q(X))$  is of type III<sub>1/q<sup>2</sup></sub>.

The work on periodic apartments was initially motivated by some of the material in [MZ2], where the authors first met the notion of doubly periodic apartments. In [Mo], S. Mozes had previously proved the abundance of doubly periodic apartments in a different context. He used them to obtain deep results on dynamical systems. We generalize the notion of doubly periodic apartments introduced in [MZ2] to that of periodic apartments and prove some surprising properties these generalizations possess. The proof of Theorem 2.3 appeared in [MZ2], but is included for the insight it provides, and Corollary 2.7 appeared as a lemma, without reference to a result analogous to Lemma 2.4.

### 1.1. Triangle Buildings

We refer the reader to [Br, R] for more details of the following facts on buildings.

A *triangle building* is a thick affine building  $\Delta$  of type  $\tilde{A}_2$ . This  $\Delta$  is a simplicial complex of rank 2 and consists of vertices, edges and triangles. We call the maximal simplices, i.e., the triangles, *chambers*. The vertices in  $\Delta$  have one of three types and every chamber contains precisely one vertex of each type. Thus there is a type map  $\tau$  defined on the vertices of  $\Delta$  such that  $\tau(v) \in \mathbb{Z}/3\mathbb{Z}$  for each vertex  $v \in \Delta$ . An automorphism  $g$  of  $\Delta$  is said to be *type-rotating* if there exists  $i \in \{0, 1, 2\}$  such that  $\tau(gv) = \tau(v) + i$  for all vertices  $v \in \Delta$ , and is said to be *type-preserving* if  $i = 0$ . We denote by  $\text{Aut}_{\text{tr}}(\Delta)$  the group of all type-rotating automorphisms of  $\Delta$ . We shall assume that each edge lies on a finite number of chambers, in which case each edge lies on precisely  $q + 1$  chambers for some integer  $q \geq 2$  and we call  $q$  the *order* of  $\Delta$ . There is a well-defined  $W$ -valued distance function on the simplices of  $\Delta$  where  $W$  is the affine Coxeter group of type  $\tilde{A}_2$ . The length function  $|\cdot|$  on  $W$  enables us to define an integer-valued distance function  $d$  on  $\Delta$ . Given a simplex  $\sigma \in \Delta$ , the *residue* or *star* determined by  $\sigma$  is the set of all simplices  $\rho$  satisfying  $d(\sigma, \rho) = 1$ .

An *apartment*  $\mathcal{A}$  in  $\Delta$  is a subcomplex isomorphic to the Coxeter complex of type  $\tilde{A}_2$ . Hence each apartment  $\mathcal{A}$  can be realized as a Euclidean plane tessellated by equilateral triangles. A *root* in  $\Delta$  is a root in any apartment  $\mathcal{A}$  of  $\Delta$ . That is to say, a root is a half plane in a geometric realization of  $\mathcal{A}$ . A *wall* in  $\Delta$  is a wall in any apartment  $\mathcal{A}$  and a *sector* is

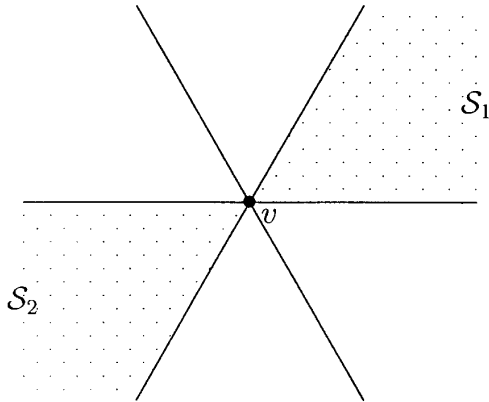
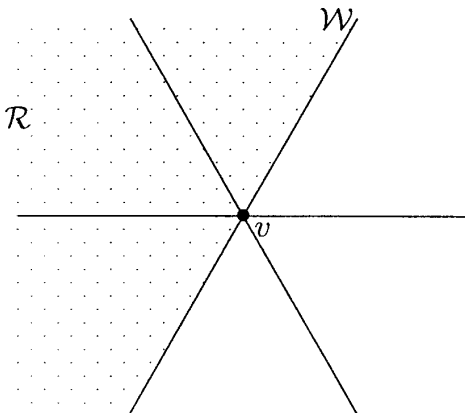


FIG. 1. Opposite sectors in an apartment.

a simplicial cone in some apartment. Two sectors are said to be *equivalent* or *parallel* if they contain a common subsector.

Given any vertex  $v$  and any chamber  $C$  in a common apartment  $\mathcal{A}$ , their convex hull in  $\mathcal{A}$  must in fact be contained in every apartment which contains both  $v$  and  $C$ . We note that given a vertex  $v \in \mathcal{A}$  we can decompose  $\mathcal{A}$  into six sectors emanating from or *based at*  $v$ . Consider two such sectors  $\mathcal{S}_1$  and  $\mathcal{S}_2$  which are opposite each other, as in Fig.1. The convex hull of  $\mathcal{S}_1$  and  $\mathcal{S}_2$  is all of  $\mathcal{A}$ , so that  $\mathcal{S}_1$  and  $\mathcal{S}_2$  uniquely determine  $\mathcal{A}$ .

*Remark 1.1.* There are an uncountable number of apartments containing any given vertex  $v \in \mathcal{A}$ . To see this, suppose that  $\mathcal{R}$  is any root

FIG. 2. A root  $\mathcal{R}$  containing the vertex  $v$ .

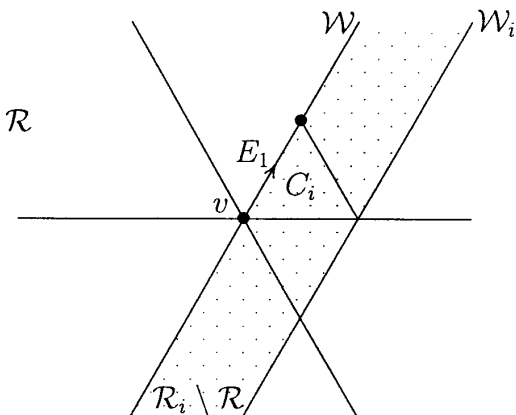


FIG. 3. Extending the root  $\mathcal{R}$  to a root  $\mathcal{R}_i$ .

containing  $v$ . Without loss of generality, suppose  $v$  is on the outside wall  $\mathcal{W}$  of  $\mathcal{R}$  as in Fig. 2.

Consider any edge on  $\mathcal{W}$ , say  $E_1$ . By thickness there exist at least two distinct chambers,  $C_1$  and  $C_2$ , incident on  $E_1$  which are not in  $\mathcal{R}$ . Adjoining either of these chambers will uniquely determine new roots  $\mathcal{R}_i$  which are the convex hulls of  $C_i$  and  $\mathcal{R}$ . Note that  $\mathcal{R}_1 \neq \mathcal{R}_2$  since they differ in at least  $C_i$ , and hence everywhere on  $\mathcal{R}_i \setminus \mathcal{R}$ . Let  $\mathcal{W}_i$  be the new outside wall of  $\mathcal{R}_i$ . See Fig. 3.

Now repeat the process for each root  $\mathcal{R}_i$ . Continuing in this manner, we obtain apartments containing the vertex  $v \in \Delta$ . The contractibility of  $\Delta$  implies that these apartments intersect only in  $\mathcal{R}$ . Using this method we have constructed at least as many apartments containing  $v$  as there are binary expansions. We therefore conclude that the number of apartments containing  $v$  is uncountable.

### 1.2. The Boundary and Its Topology

Let  $\Omega$  be the set of equivalence classes of sectors in  $\Delta$ . Given any  $\omega \in \Omega$  and any fixed vertex  $v$  there is a unique sector  $\mathcal{S}_v(\omega)$  in the equivalence class  $\omega$  based at  $v$ . The unique chamber  $C \in \mathcal{S}_v(\omega)$  containing  $v$  is called the *base chamber* of  $\mathcal{S}_v(\omega)$ . Suppose that  $\mathcal{S}_v(\omega)$  has base chamber  $C$ . For each vertex  $u \in \mathcal{S}$ , we denote the convex hull of  $u$  and  $C$  by  $I_v^u(\omega)$ . We illustrate such a convex hull  $I_v^u(\omega)$  as a shaded region in Fig. 4.

$\Omega$  is in fact the set of chambers of the spherical building  $\Delta^\infty$  at infinity associated to  $\Delta$ . We shall refer to  $\Omega$  as the *boundary* of  $\Delta$ . This boundary is a totally disconnected compact Hausdorff space with a base for the topology consisting of sets of the form

$$U_v^u(\omega) = \{ \omega' \in \Omega : I_v^u(\omega) \subseteq \mathcal{S}_v(\omega') \},$$

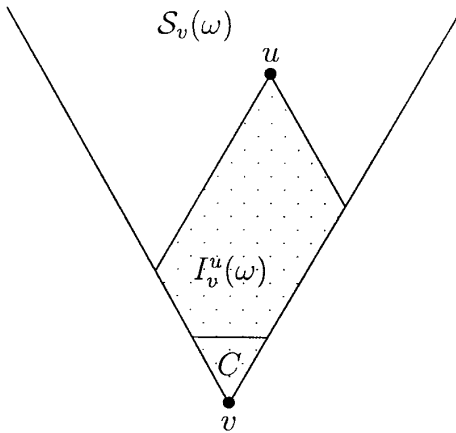


FIG. 4. Region  $I_v^u(\omega)$  of  $\mathcal{S}_v(\omega)$ .

where  $v$  is an arbitrary vertex,  $\omega \in \Omega$ , and  $u$  is any vertex in  $\mathcal{S}_v(\omega)$  [CMS]. Note that  $U_v^u(\omega') = U_v^u(\omega)$  whenever  $\omega' \in U_v^u(\omega)$ . The sets  $U_v^u(\omega)$  form a basis of open and closed sets for the topology on  $\Omega$ . This topology is independent of the vertex  $v$  [CMS, Lemma 2.5].

### 1.3. Triangle Groups

Let  $(P, L)$  be a finite projective plane of order  $q$ . Thus there are  $|P| = q^2 + q + 1$  points and  $|L| = q^2 + q + 1$  lines with each point lying on  $q + 1$  lines and each line having  $q + 1$  points lying on it. Let  $\lambda: P \rightarrow L$  be a point-line correspondence, i.e., a bijection between the elements of  $P$  and those of  $L$ .

Let  $\mathcal{T}$  be a set of triples  $\{(x, y, z): x, y, z \in P\}$  satisfying the following properties:

- (1) For all  $x, y \in P$ ,  $(x, y, z) \in \mathcal{T}$  for some  $z \in P \Leftrightarrow y$  and  $\lambda(x)$  are incident.
- (2)  $(x, y, z) \in \mathcal{T} \Rightarrow (y, z, x) \in \mathcal{T}$ .
- (3) For all  $x, y \in P$ ,  $(x, y, z) \in \mathcal{T}$  for at most one  $z \in P$ .

Such a set  $\mathcal{T}$  is called a *triangle presentation*. The notion of triangle presentations was introduced and developed in [CMSZ] and the reader is referred to these papers for proofs of statements regarding triangle presentations quoted below without reference.

Let  $\{a_x: x \in P\}$  be a set of  $q^2 + q + 1$  distinct letters and define a multiplicative group  $\Gamma$ , whose identity shall be denoted by  $e$ , with the following presentation:

$$\Gamma = \langle a_x: a_x a_y a_z = e \text{ for all } (x, y, z) \in \mathcal{T} \rangle.$$

The Cayley graph of  $\Gamma$  constructed via right multiplication with respect to the generators  $a_x, x \in P$  and their inverses is in fact the skeleton of a triangle building  $\Delta_{\mathcal{T}}$  whose chambers can be identified with the sets  $\{g, ga_x^{-1}, ga_y\}$  for  $g \in \Gamma$  and where  $(x, y, z) \in \mathcal{T}$  for some  $z \in P$ . Furthermore,  $\Gamma$  acts simply transitively by left multiplication on the vertices of  $\Delta_{\mathcal{T}}$  in a type-rotating manner. Henceforth we assume that  $\Delta = \Delta_{\mathcal{T}}$  for some triangle presentation  $\mathcal{T}$ .

Given a sector  $\mathcal{S}(\omega) \subset \Delta$ , left multiplication of every vertex in  $\mathcal{S}(\omega)$  by an element  $g \in \Gamma$  defines an action of  $\Gamma$  on  $\Omega$ . We refer the reader to [CMS] for a proof that this action is well-defined.

The boundary  $\Omega$  can be expressed as a union of  $(q^2 + q + 1)(q + 1)$  disjoint sets since, for any fixed vertex  $v$ ,

$$\Omega = \bigcup_{\substack{\text{chambers } C \\ \text{with } v \in C}} \Omega_v^C$$

where  $\Omega_v^C$  denotes the set of  $\omega \in \Omega$  whose representative sector based at  $v$  has base chamber  $C$ .

For brevity we will denote the region  $I_v^u(\omega)$  simply by  $I^u(\omega)$ .

#### 1.4. Labelling Apartments with Elements of $\Gamma$

Suppose we are given a triangle group  $\Gamma$  and its corresponding triangle building  $\Delta$ . Each of the edges in  $\Delta$  can be labelled by a generator or an

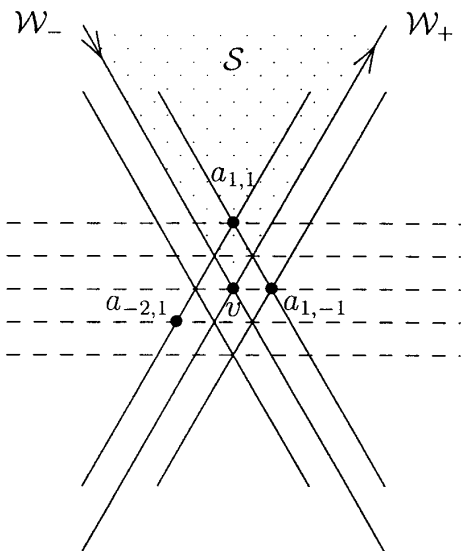


FIG. 5. Labelling of an apartment with respect to a sector  $\mathcal{S}$ .

inverse generator of  $\Gamma$ . By providing each edge with an orientation, it suffices to label the edges with generators of  $\Gamma$ . For each  $g \in \Gamma$  and each  $x \in P$ , we label the edge  $g \rightarrow ga_x$  in  $\Delta$  by  $a_x$  or more briefly by  $x$  if there is no likelihood of confusion.

Suppose we are given an apartment  $\mathcal{A}$  and a sector  $\mathcal{S} \subset \mathcal{A}$  with base vertex  $v$ . Each sector wall of  $\mathcal{S}$  is half of a wall in  $\mathcal{A}$ . Each such wall consists entirely of edges with the same orientation. Let  $\mathcal{W}_+$  be the wall containing the sector panel whose orientation emanates from  $v$ , and denote the other sector wall of  $\mathcal{S}$  by  $\mathcal{W}_-$ . The walls  $\mathcal{W}_+$  and  $\mathcal{W}_-$  and their translates in  $\mathcal{A}$  form a lattice in  $\mathcal{A}$ . Thus each vertex in  $\mathcal{A}$  can be given a coordinate with respect to this lattice where the sector panels of  $\mathcal{S}$  are taken to be in the positive direction and the coordinate in the direction of  $\mathcal{W}_+$  is given first. Hence  $\mathcal{A}$  can be determined via its vertices as  $\mathcal{A} = (a_{i,j})_{i,j \in \mathbb{Z}}$  where each  $a_{i,j} \in \Gamma$ . See Fig. 5 for an example.

We note that this induces a labelling of  $\mathcal{S}$  which is independent of the particular apartment  $\mathcal{A}$  provided that  $\mathcal{S} \subset \mathcal{A}$ .

1.5. *Borel Probability Measures on  $\Omega$*

We refer the reader to [CM] and [CMS] for details of the results quoted in this section. For each vertex  $v \in \Delta$ , we denote by  $V_v^{m,n}$  the set of vertices  $u \in \Delta$  for which there exists a sector  $\mathcal{S}_v$  based at  $v$  such that  $u = a_{m,n}$  with respect to the labelling of  $\mathcal{S}_v$  described in Section 1.4. Thus  $d(v, u) = m + n$  for all  $u \in V_v^{m,n}$ . The cardinalities  $N_{m,n} = |V_v^{m,n}|$  are independent of  $v$  and are given by

$$N_{m,n} = \begin{cases} (q^2 + q + 1)(q^2 + q) q^{2(m+n-2)} & \text{if } m, n \geq 1, \\ (q^2 + q + 1) q^{2(m-1)} & \text{if } m \geq 1, n = 0, \\ (q^2 + q + 1) q^{2(n-1)} & \text{if } n \geq 1, m = 0, \\ 1 & \text{if } m = 0 = n. \end{cases}$$

For every pair of vertices  $v, u \in \Delta$  we define

$$\Omega_v^u = \{ \omega \in \Omega : u \in \mathcal{S}_v(\omega) \}.$$

Thus, in terms of the open sets described in Section 1.2,

$$\Omega_v^u = \bigcup_{\substack{\omega \in \Omega \\ u \in \mathcal{S}_v(\omega)}} U_v^u(\omega)$$

although there is only ever one distinct set  $U_v^u(\omega)$  contributing to this union unless  $v$  and  $u$  lie on a common wall in  $\Delta$ . If  $v$  and  $u$  lie on a common wall in  $\Delta$ , there are precisely  $q + 1$  distinct and disjoint sets of the form  $U_v^u(\omega)$  in the above union.

It was noted in [CMS] that, for each vertex  $v \in \mathcal{A}$ , there exists a natural Borel probability measure  $\nu_v$  on  $\Omega$  which assigns equal measure to each of the  $N_{m,n}$  disjoint sets  $\Omega_v^u$  for  $u \in V_v^{m,n}$ . Furthermore, given any two vertices  $u, v \in \mathcal{A}$ , the measures  $\nu_u$  and  $\nu_v$  are mutually absolutely continuous [CMS, Lemma 2.5]. We define  $\nu = \nu_e$  for brevity and denote  $L^\infty(\Omega, \nu)$  by  $L^\infty(\Omega)$ .

For any  $g \in \Gamma$  and any vertex  $v \in \mathcal{A}$ ,  $\nu_{gv} = g\nu_v$  in the sense that  $\nu_{gv}(E) = \nu_v(g^{-1}E)$  for any Borel set  $E \subseteq \Omega$ . Since  $\nu$  is therefore quasi-invariant under the action of  $\Gamma$ , the von Neumann algebra  $L^\infty(\Omega) \rtimes \Gamma$  is well-defined.

## 2. PERIODICITY IN BUILDINGS

### 2.1. Periodic Apartments

Suppose that we have an apartment  $\mathcal{A}$  labelled with respect to some sector  $\mathcal{S} \subset \mathcal{A}$  as described in Section 1.4.

DEFINITIONS 2.1. We define the *lattice of periodicity points*,  $\mathcal{L}$ , of  $\mathcal{A}$  by

$$\mathcal{L} = \{(r, s) \in \mathbb{Z}^2 : a_{i,j}^{-1}a_{k,l} = a_{i+r,j+s}^{-1}a_{k+r,l+s} \text{ for all } i, j, k, l \in \mathbb{Z}\}.$$

To appreciate the meaning of  $\mathcal{L}$ , note that there is a natural action by translation of  $\mathbb{Z}^2$  on the apartment  $\mathcal{A}$  given by  $(r, s)a_{i,j} = a_{i+r,j+s}$ . Suppose  $a_{i,j}$  and  $a_{k,l}$  are adjacent vertices. The element  $a_{i,j}^{-1}a_{k,l}$  defines a labelled edge from  $a_{i,j}$  to  $a_{k,l}$ . If  $(r, s) \in \mathcal{L}$  then the element  $a_{i+r,j+s}^{-1}a_{k+r,l+s}$  defines exactly the same labelling on the edge from  $a_{i+r,j+s}$  to  $a_{k+r,l+s}$ . Thus a translation by  $(r, s) \in \mathcal{L}$  leaves the edge labelling of the apartment invariant.

Note that  $\mathcal{L}$  is a subgroup of  $\mathbb{Z}^2$ , so that we must have either  $\mathcal{L} = \{(0, 0)\}$ ,  $\mathcal{L} \cong \mathbb{Z}$  or  $\mathcal{L} \cong \mathbb{Z}^2$ . We call the elements  $(r, s) \in \mathcal{L}$  the *periodicity points* of  $\mathcal{A}$ .

We say that  $\mathcal{A}$  is *periodic* if  $\mathcal{L}$  is non-trivial. We distinguish between the cases  $\mathcal{L} \cong \mathbb{Z}$  and  $\mathcal{L} \cong \mathbb{Z}^2$  by saying that  $\mathcal{A}$  is *singly periodic* or *doubly periodic*, respectively. We say that  $\mathcal{A}$  is  $(r, s)$ -*periodic* if  $(r, s) \in \mathcal{L}$  regardless of whether  $\mathcal{A}$  is singly or doubly periodic. In geometric terms this means that the labelling of the directed edges of  $\mathcal{A}$  by generators of  $\Gamma$  has a translational symmetry in the  $(r, s)$  direction. For brevity we say  $\mathcal{A}$  is  $\{(r, s), (t, u)\}$ -*periodic* if  $\mathcal{A}$  is both  $(r, s)$ -periodic and  $(t, u)$ -periodic.

We note that the set of periodic apartments in  $\mathcal{A}$  is  $\Gamma$ -invariant since if  $\mathcal{A}$  is periodic then  $g\mathcal{A}$  is necessarily periodic and has the same lattice of periodicity points for all  $g \in \Gamma$ .



Given a periodic apartment  $\mathcal{A}$  with periodicity lattice  $\mathcal{L}$ , define  $m \in \mathbb{Z}$  by

$$m = \min_{\substack{r,s \in \mathcal{L} \\ (r,s) \neq (0,0)}} |a_{i,j}^{-1} a_{i+r,j+s}|$$

for any  $i, j \in \mathbb{Z}$ . Such an integer exists and is positive since the length of a word in  $\Gamma$  is a positive integer and  $\mathcal{L}$  is non-trivial since we are assuming periodicity. We say that  $\mathcal{A}$  has *minimal period*  $m$ .

## 2.2. Rigidly Periodic Apartments

We introduce some periodic apartments whose behaviour is somewhat special.

**DEFINITION 2.2.** Call an apartment  $\mathcal{A}$  *rigidly periodic* if either

- $\mathcal{A}$  is doubly periodic, or
- $\mathcal{A}$  is  $(r, s)$ -periodic with  $r, s \neq 0$  and  $s \neq -r$ .

Thus singly periodic apartments whose vertices  $a_{r,s}$ ,  $(r, s) \in \mathcal{L}$  did not lie entirely along a single wall in the apartment would be rigidly periodic. In fact, it turns out that rigid and double periodicity are equivalent notions, see Lemma 2.10.

Note that if  $\mathcal{A}$  is rigidly periodic then so is  $g\mathcal{A}$  for all  $g \in \Gamma$  so that the set of rigidly periodic apartments is  $\Gamma$ -invariant. Since  $\Gamma$  is countable, there can only be a countable number of rigidly periodic apartments containing a fixed vertex  $v \in \mathcal{A}$ . Recall that there is an uncountable number of apartments containing  $v$  by Remark 1.1. We deduce that, for any vertex  $v \in \mathcal{A}$ , there exist apartments containing  $v$  which are not rigidly periodic.

The existence of rigidly periodic apartments with arbitrarily large minimal periodicity was established by Mozes [Mo, Theorem 2.2'] who showed that periodic apartments are dense in the case where  $\Gamma$  embeds as a cocompact lattice in a strongly transitive group of automorphisms of  $\mathcal{A}$ . The existence of rigidly periodic apartments in the context of  $\tilde{A}_2$  groups was established by Mantero and Zappa in [MZ2, Lemma 2.4 and Proposition 2.5]. We outline their proof since it is constructive and therefore more useful than the mere existence result and we refer the reader to [MZ2] for the full details.

**THEOREM 2.3.** *There exist rigidly periodic apartments with arbitrarily large minimal period.*

*Proof.* Suppose we wish to construct a rigidly periodic apartment with minimal period greater than a fixed positive integer  $m$ . We begin by fixing a non-periodic apartment  $\mathcal{A}$  containing  $e$  which is labelled with respect to

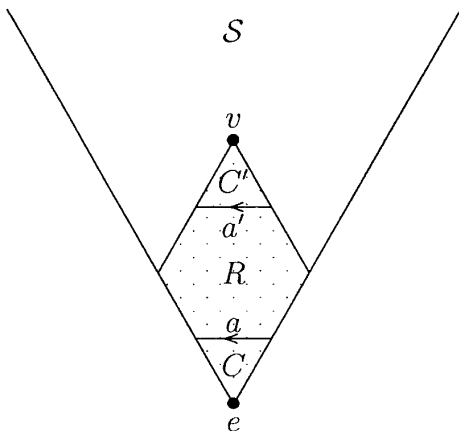


FIG. 6. Convex hull  $R$  of  $C$  and  $v$ .

some sector  $\mathcal{S}$  based at  $e$ . Thus  $\mathcal{A} = (a_{i,j})$  with  $a_{0,0} = e$ . Denote by  $C$  the base chamber of  $\mathcal{S}$ . Let  $v = a_{m,m} \in \mathcal{A}$  and denote by  $R$  the convex hull of  $C$  and  $v$  in  $\mathcal{A}$ .  $R$  is a diamond with the vertices  $e$  and  $v$  at opposite corners. Let  $C'$  be the unique chamber in  $R$  containing  $v$ .

Suppose that the edge on  $C$  opposite  $e$  is labelled  $a$ , and that the edge on  $C'$  opposite  $v$  is labelled  $a'$ ; see Fig. 6.

By altering our choice of non-periodic apartment  $\mathcal{A}$  if necessary, we may assume that  $a \neq a'$  and that  $R$  is not contained in a rigidly periodic apartment with minimal period less than  $m$ .

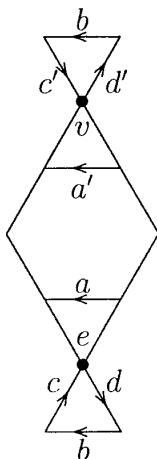


FIG. 7. The region  $R'$ .

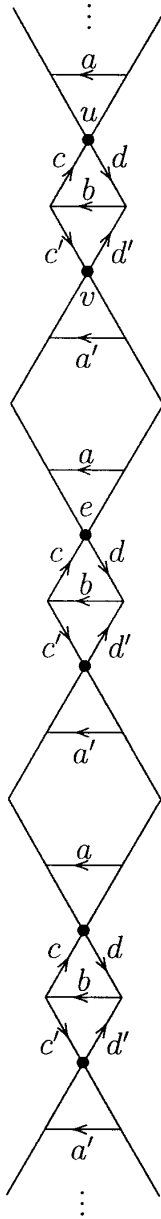


FIG. 8. Infinite tessellation of the region  $R'$ .

Results in [MZ2] prove that there exist chambers  $D$  and  $D'$  satisfying the following conditions:

- (1)  $D$  is opposite  $C$  in the residue determined by  $e$ .
- (2)  $D'$  is opposite  $C'$  in the residue determined by  $v$ .
- (3)  $D$  corresponds to a relation  $bcd = e$  and the edge labelled  $b$  is opposite the vertex  $e$ .
- (4)  $D'$  corresponds to a relation  $bc'd' = e$  and the edge labelled  $b$  is opposite the vertex  $v$ .
- (5)  $c \neq c'$  and  $d \neq d'$ .

Thus we can extend  $R$  non-trivially at both ends by  $D$  and  $D'$  to construct a region  $R'$ , pictured in Fig. 7, which undergoes no cancellation upon iteration. This enables us to construct an apartment  $\mathcal{A}'$  containing the region obtained by tessellating  $R'$  infinitely in the directions of  $\mathcal{S}$  and its opposite as in Fig. 8.

The convex hull of the infinite tessellation of  $R'$  uniquely determines the entire apartment  $\mathcal{A}'$ . Let

$$u = a_{m+2, m+2} = vd'd^{-1} = v(c')^{-1}c$$

as labelled on Fig. 8. Note that the labelling of the edges of  $\mathcal{A}'$  looks the same when viewed from each of the vertices  $u^n \in \mathcal{A}'$  and hence  $u^n \mathcal{A}' = \mathcal{A}'$  for every  $n \in \mathbb{Z}$ . Thus  $\mathcal{A}'$  is  $(m+2, m+2)$ -periodic with minimal periodicity greater than or equal to  $m$  by construction and its periodicity lattice will contain the elements  $(n(m+2), n(m+2))$  for all  $n \in \mathbb{Z}$ . ■

The following observation enables us to deduce some rather nice properties of rigidly periodic apartments and the boundary points they define.

LEMMA 2.4. *A rigidly periodic apartment is completely determined by any single sector.*

*Proof.* We begin by labelling the apartment  $\mathcal{A}$  with respect to the known sector  $\mathcal{S}$  so that the set of vertices  $\{a_{k,l} : k, l \geq 0\}$  is known. We show that in fact  $a_{i,j}$  is then uniquely determined for all  $i, j \in \mathbb{Z}$ . Without loss of generality, we consider three separate cases.

*Case 1. The doubly periodic case.* Suppose that  $\mathcal{A}$  is an  $\{(r, s), (t, u)\}$ -periodic apartment with  $(r, s)$  and  $(t, u)$  being linearly independent vectors in  $\mathbb{R}^2$ . Thus, given any  $i, j \in \mathbb{Z}$ , there exist  $m, n \in \mathbb{Z}$  such that  $mr + nt, ms + nu, i + mr + nt$ , and  $j + ms + nu$  are all positive. From the properties of double periodicity, we deduce that

$$a_{i,j} = a_{k,l} a_{k+mr+nt, l+ms+nu}^{-1} a_{i+mr+nt, j+ms+nu}$$

so that if  $k, l \geq 0$  every term on the right hand side of the equation is a vertex in  $\mathcal{S}$  and is therefore known. Since  $i$  and  $j$  were arbitrary integers,  $a_{i,j}$  is uniquely determined for all  $i, j \in \mathbb{Z}$ . We deduce that  $\mathcal{A}$  is completely determined.

*Case 2. The  $(r, s)$ -periodic case with  $r, s > 0$ .* In this case given any  $i, j \in \mathbb{Z}$  there exists a positive integer  $m$  such that  $i + mr, j + ms > 0$ . Once again this leads to an equation of the form

$$a_{i,j} = a_{k,l} a_{k+mr, l+ms}^{-1} a_{i+mr, j+ms},$$

where every element on the right hand side is a vertex in  $\mathcal{S}$  and therefore uniquely determined.

*Case 3. The  $(r, s)$ -periodic case with either  $r > 0$  but  $s < 0$  or  $r < 0$  but  $s > 0$ .* We note that the vertices  $a_{mr, ms}$  are determined for all  $m \in \mathbb{Z}$  since

$$a_{mr, ms} = a_{k+mr, l+ms} a_{k,l}^{-1} a_{0,0}$$

and suitably large choices of  $k, l$  ensure that all the elements on the right hand side are vertices in  $\mathcal{S}$ . Since  $r \neq 0 \neq s \neq -r$  the vertex  $a_{mr, ms}$  does not lie on a sector wall emanating from  $a_{0,0}$ . Suppose that  $\mathcal{S}$  has base chamber  $C$ . The convex hull of  $C$  and  $a_{mr, ms}$  for all  $m \in \mathbb{Z}$  must also be contained in  $\mathcal{A}$ . By taking arbitrarily large positive values for  $m$  we know that a sector  $\mathcal{S}'$  adjacent to  $\mathcal{S}$  is determined. Taking arbitrarily large negative values for  $m$  implies that the sector  $\mathcal{S}''$  opposite  $\mathcal{S}'$  is determined. Since the convex hull of  $\mathcal{S}'$  and  $\mathcal{S}''$  is  $\mathcal{A}$  this means that  $\mathcal{A}$  is entirely determined. Figure 9 illustrates the proof in the case of  $r > 0$  and  $s < 0$  where the shaded area represents the convex hull of  $C$  and  $a_{mr, ms}$ .

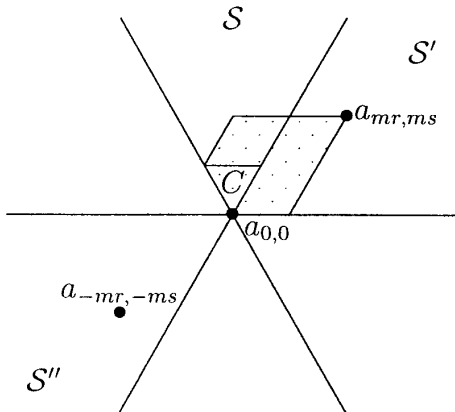


FIG. 9. Illustration of the proof of case 3 with  $r > 0$  and  $s < 0$ .

This completes the proof since all other cases are equivalent to one of those considered. ■

**COROLLARY 2.5.** *Suppose an apartment  $\mathcal{A} = (a_{i,j})_{i,j \in \mathbb{Z}}$  is rigidly periodic. If  $\mathcal{A}$  is  $(r, s)$ -periodic for some  $r, s \in \mathbb{Z}$  with  $r, s \neq 0$  and  $s \neq -r$  then  $\mathcal{A}$  is uniquely determined by  $a_{i,j}$ ,  $a_{i+r,j+s}$  and  $a_{i,j}^{-1}a_{i+r,j+s}$  for every  $i, j \in \mathbb{Z}$ .*

*Proof.* Let  $g = a_{i,j}^{-1}a_{i+r,j+s}$  and  $\mathcal{C}$  be the convex hull of  $a_{i,j}$  and  $a_{i+r,j+s}$  as in Fig. 10. We may suppose that  $e \in \mathcal{A}$ .

Then  $g^n \mathcal{A} = \mathcal{A}$  for every  $n \in \mathbb{Z}$  and the convex hull of the iterates  $g^n \mathcal{C}$ , for  $n \in \mathbb{N}$  determine a sector  $\mathcal{S}$  in  $\mathcal{A}$ . Thus  $\mathcal{A}$  is uniquely determined by Lemma 2.4. ■

**COROLLARY 2.6.** *A boundary point  $\omega \in \Omega$  can have representative sectors in at most one rigidly periodic apartment  $\mathcal{A}$ .*

*Proof.* Suppose  $\mathcal{A}$  and  $\mathcal{A}'$  are both rigidly periodic apartments which contain representative sectors of  $\omega$ ; say  $\mathcal{S} \subset \mathcal{A}$  and  $\mathcal{S}' \subset \mathcal{A}'$  are two such sectors. Since  $\mathcal{S}$  and  $\mathcal{S}'$  are equivalent sectors, they must contain a common subsector  $\mathcal{S}''$ . Hence  $\mathcal{S}'' \subseteq \mathcal{A} \cap \mathcal{A}'$ , and therefore by Lemma 2.4 we must have  $\mathcal{A} = \mathcal{A}'$ . ■

The following corollary is a generalization of [MZ2, Lemma 3.4].

**COROLLARY 2.7.** *Let  $\mathcal{A}$  and  $\mathcal{A}'$  be rigidly periodic apartments which contain representative sectors of  $\omega, \omega' \in \Omega$  respectively. If  $g\omega = \omega'$  for some  $g \in \Gamma$  then  $\mathcal{A}' = g\mathcal{A}$ . In particular, if  $\mathcal{A} = \mathcal{A}'$  then  $g$  must stabilize  $\mathcal{A}$ .*

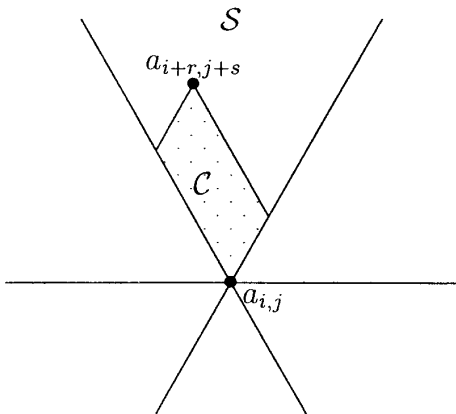


FIG. 10. Convex hull  $\mathcal{C}$  of  $a_{i,j}$  and  $a_{i+r,j+s}$ .

*Proof.* Suppose  $\mathcal{S}_v(\omega) \subset \mathcal{A}$  is a representative of  $\omega$ . The apartment  $g\mathcal{A}$  is rigidly periodic and contains  $g\mathcal{S}_v(\omega) = \mathcal{S}_{gv}(\omega) = \mathcal{S}_{gv}(\omega')$ , a representative of  $\omega'$ . By Corollary 2.6 we must have  $\mathcal{A}' = g\mathcal{A}$ . ■

Notice that it is not necessary to know *a priori* that  $\mathcal{A}$  and  $\mathcal{A}'$  have the same minimal period; we simply need to know that they are both rigidly periodic.

If we know that an apartment is periodic and that it is stabilized by a certain subset of  $\Gamma$ , the following result enables us to bound the minimal periodicity of  $\mathcal{A}$ .

**PROPOSITION 2.8.** *Let  $F \subset \Gamma \setminus \{e\}$  be a finite set, and suppose  $\mathcal{A}$  is a periodic apartment such that  $g\mathcal{A} = \mathcal{A}$  for all  $g \in F$ . Then  $\mathcal{A}$  can have minimal period of at most  $2 \max\{|g| : g \in F\}$ .*

*Proof.* Given such a periodic apartment  $\mathcal{A}$ , label its vertices with respect to some sector  $\mathcal{S} \in \mathcal{A}$  and suppose that  $\mathcal{A}$  has minimal period  $m$ . Denote by  $S$  the group of symmetries generated by reflections in  $\mathcal{A}$  fixing  $a_{00}$ . We begin by noting that if  $g\mathcal{A} = \mathcal{A}$  then  $ga_{i,j} = a_{k,l}$  where

$$(k, l) = T_g(i, j) = \sigma(i, j) + (r, s)$$

for some  $\sigma \in S$  and  $r, s \in \mathbb{Z}$ . This is a slight generalization of a statement made in [MZ2] and we refer the reader to that paper for a proof since it generalizes in a very straightforward manner.

Since  $T_g$  corresponds to a non-trivial symmetry of  $\mathcal{A}$ ,  $(r, s)$  must be non-trivial whenever  $\sigma$  is; otherwise the symmetry of  $\mathcal{A}$  would be a reflection and would collapse  $\mathcal{A}$  to a root. Thus each  $g \in F$  which stabilizes  $\mathcal{A}$  must either induce a translation or a glide reflection on  $\mathcal{A}$ . If  $g$  induces a translation on  $\mathcal{A}$  then  $|g| \geq m$ . On the other hand, if  $g$  induces a glide-reflection on  $\mathcal{A}$  then  $g^2$  must induce a translation on  $\mathcal{A}$  so that  $|g^2| \geq m$ . Since  $2|g| \geq |g^2|$  the result follows. ■

**COROLLARY 2.9.** *Given a finite set  $F \subset \Gamma \setminus \{e\}$ , there exists a positive integer  $m(F)$  such that no element of  $F$  can stabilize a periodic apartment with minimal period greater than  $m(F)$ .*

If  $F = \{g\}$  we denote  $m(F)$  by  $m(g)$ .

The following result is due to Mozes (cf. proof of Proposition 2.13 of [Mo]) and was communicated to us by Steger. We include it only for the benefit of the curious since we will not need to apply the result.

**LEMMA 2.10.** *Every rigidly periodic apartment is doubly periodic.*

*Proof.* We need to show that every apartment  $\mathcal{A}$  which is  $(r, s)$ -periodic with  $r, s \neq 0$  and  $s \neq -r$  is in fact doubly periodic. We begin by noting that, with a suitable relabelling of  $\mathcal{A}$  we may assume  $r, s > 0$ .

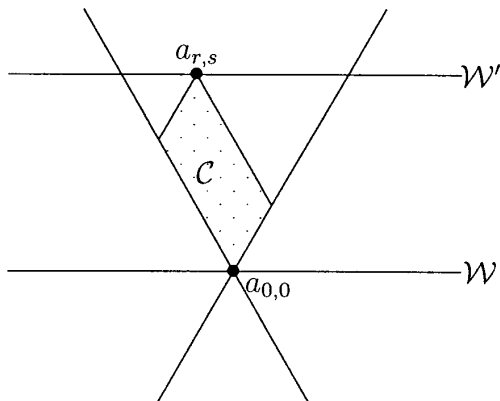


FIG. 11. Strip between walls  $\mathcal{W}$  and  $\mathcal{W}'$  showing convex hull  $\mathcal{C}$  of  $a_{0,0}$  and  $a_{r,s}$ .

By Corollary 2.5, the apartment  $\mathcal{A}$  and its labelling is completely determined by its periodicity and by the convex hull,  $\mathcal{C}$ , of  $a_{0,0}$  and  $a_{r,s}$ . Consider the strip of the apartment between the wall  $\mathcal{W}$  containing the vertices  $a_{m,-m}$ , and the parallel wall  $\mathcal{W}'$  containing the vertices  $a_{r+m,s-m}$ . This strip is depicted in Fig. 11, where the shaded area represents  $\mathcal{C}$ . We note that  $\mathcal{C}$  contains only a finite number of chambers.

The convex hull of the vertices  $a_{m,-m}$  and  $a_{m+r,-m+s}$  will have the same shape as  $\mathcal{C}$  for every  $m \in \mathbb{Z}$ . We consider the labelling of these convex hulls. There are only a finite number of possible labellings for a chamber and hence there are only a finite number of ways these convex hulls can be labelled. However, the strip in question is of infinite length so there must be at least two such convex hulls which share the same labelling. Without loss of generality we may assume that one of these is  $\mathcal{C}$ . Suppose that the other is the convex hull  $\mathcal{C}'$  of  $a_{n,-n}$  and  $a_{n+r,-n+s}$  for some  $n \in \mathbb{Z}$ .

The  $(r, s)$ -periodicity and  $\mathcal{C}'$  completely determine  $\mathcal{A}$ , as was the case for  $\mathcal{C}$ . Moreover the labelling of  $\mathcal{A}$  induced by this construction is identical to that produced by  $\mathcal{C}$ , but shifted by  $(n, -n)$ . Hence  $\mathcal{A}$  must be  $(n, -n)$ -periodic as well as rigidly periodic. Thus  $\mathcal{A}$  is doubly periodic. ■

Lemma 2.10 therefore establishes the equivalence of rigid periodicity and double periodicity.

### 2.3. Periodic Walls, Sectors, and Roots

We note that the notions of periodic apartments and minimal periodicity can be generalized so that they apply to walls, sectors and roots in  $\mathcal{A}$  by imposing various conditions on the integers  $i, j, k, l, i+r, j+s, k+r, l+s$  in the equation

$$a_{i,j}^{-1} a_{k,l} = a_{i+r,j+s}^{-1} a_{k+r,l+s}.$$



The set of pairs  $(r, s) \in \mathbb{Z}^2$  satisfying such an equation is then no longer necessarily a subgroup of  $\mathbb{Z}^2$ .

Without loss of generality, we may assume that the vertex  $a_{0,0}$  is on the wall, or is the base vertex of the sector, or is on the boundary of the root in question. We generalize the notions of rigid periodicity and single periodicity to the context of sectors and roots.

It is not in general true that a periodic root extends to a periodic apartment or that such an apartment is unique if it exists. However, the arguments used in the proof of Lemma 2.4 show that a rigidly periodic sector determines a unique rigidly periodic apartment. We complete the analysis of periodic sectors with the following result.

LEMMA 2.11. *A periodic sector which is not rigidly periodic determines a unique periodic root.*

*Proof.* Suppose that  $\mathcal{S}$  is a periodic sector whose vertices are labelled with respect to its base vertex,  $a_{0,0}$  in such a way that  $a_{i,j} \in \mathcal{S}$  for all  $i, j \in \mathbb{N}$ . We may suppose  $a_{0,0} = e$ . Suppose also that  $\mathcal{S}$  is not rigidly periodic.

If  $\mathcal{S}$  is  $(r, 0)$ -periodic or  $(0, s)$ -periodic for some  $r$  or  $s \in \mathbb{Z}$  let  $g = a_{r,0}$  or  $g = a_{0,s}$  respectively. If  $\mathcal{S}$  is  $(r, -r)$ -periodic for some  $r \in \mathbb{N}$ , let  $g = a_{r,-r}$ .

The sectors  $g^n \mathcal{S}$  for  $n \in \mathbb{Z}$  then lie in a common apartment since the labelling of the directed edges in  $\mathcal{S}$  by generators of  $\Gamma$  have a symmetry in the required direction. There may be several apartments containing the sectors  $g^n \mathcal{S}$  for  $n \in \mathbb{Z}$ . However the convex hull of the sectors  $g^n \mathcal{S}$  for  $n \in \mathbb{Z}$  in any apartment containing them will determine a unique periodic root  $\mathcal{R}$ . In particular, the wall bounding  $\mathcal{R}$  and its translates in  $\mathcal{R}$  are periodic. ■

### 3. PERIODIC LIMIT POINTS AND THEIR PROPERTIES

#### 3.1. Periodic Limit Points and Their Behaviour under the Action of $\Gamma$

We begin by using the notion of periodicity in  $\Delta$  to define a useful subset of  $\Omega$ .

DEFINITIONS 3.1. Call a boundary point  $\omega \in \Omega$  a *periodic limit point* if it has a periodic representative sector  $\mathcal{S}(\omega)$ , say. We refer to such a sector  $\mathcal{S}(\omega)$  as a *periodic representative* of  $\omega$ . Note that we do not assume that every periodic limit point has a periodic representative  $\mathcal{S}(\omega)$  with  $e \in \mathcal{S}(\omega)$ . We denote the set of periodic limit points by  $\Pi$ .

A most intriguing property of periodic limit points, proved forthwith, is that they are the only boundary points which can be stable under the action of an element of  $\Gamma$ .

PROPOSITION 3.2. *Suppose  $\omega \in \Omega$  satisfies  $g\omega = \omega$  for some  $g \in \Gamma \setminus \{e\}$ . Then we must have  $\omega \in \Pi$ .*

*Proof.* Suppose  $\omega \in \Omega$  satisfies  $g\omega = \omega$  for some  $g \in \Gamma$ . Then  $g^n\omega = \omega$  for all  $n \in \mathbb{Z}$ . Let  $\mathcal{S} = \mathcal{S}(\omega)$  be any representative sector of  $\omega$ . Thus  $g^n\mathcal{S}$  is a representative of  $\omega$  for every  $n \in \mathbb{Z}$ . Consider first the representatives  $\mathcal{S}$  and  $g\mathcal{S}$  of  $\omega$ . Note that all the vertices in  $g\mathcal{S}$  are of the form  $gv$  where  $v \in \Gamma$  is a vertex in  $\mathcal{S}$ .

Since  $\mathcal{S}$  and  $g\mathcal{S}$  are parallel they must contain a common subsector. Suppose  $\mathcal{S}_{gv} = \mathcal{S}_{gv}(\omega) \subseteq \mathcal{S} \cap g\mathcal{S}$  is such a common subsector. Since  $\mathcal{S}_{gv}(\omega) \subseteq g\mathcal{S}$ , we must have  $g^{-1}\mathcal{S}_{gv}(\omega) \subseteq \mathcal{S}$ , which is to say  $\mathcal{S}_v(\omega) \subseteq \mathcal{S}$ . Thus the sectors  $\mathcal{S}_v = \mathcal{S}_v(\omega)$  and  $g\mathcal{S}_v = \mathcal{S}_{gv}$  are both contained in  $\mathcal{S}$  and hence are contained in a common apartment which we may assume contains  $e$ . Since  $\mathcal{S}_v$  and  $g\mathcal{S}_v$  are parallel they must intersect in a sector which is a translate of each of them. Thus the labelling of  $\mathcal{S}_v$  must have a  $g$ -translational symmetry. The sector  $\mathcal{S}_v$  is therefore a periodic representative of  $\omega$  and hence  $\omega \in \Pi$ , thus establishing the claim. ■

DEFINITIONS 3.3. If  $\omega \in \Omega$  has a periodic representative  $\mathcal{S}(\omega)$  which is in fact rigidly periodic, Lemma 2.4 and Corollary 2.6 enable us to associate to  $\omega$  a unique rigidly periodic apartment  $\mathcal{A}(\omega)$ . In this case we say that  $\omega$  is a *rigidly periodic limit point* and we denote by  $\Lambda$  the subset of  $\Omega$  consisting of all rigidly periodic limit points.

Remark 3.4. Given a particular  $g \in \Gamma$  there are at most a countable number of rigidly periodic apartments containing  $g$ . Since  $\Gamma$  is countable this means that there are at most a countable number of distinct rigidly periodic apartments in  $\Lambda$  and therefore  $\Lambda$  is a countable subset of  $\Omega$ . Note that  $\Omega$  is uncountable by an argument analogous to that of Remark 1.1. There must therefore exist boundary points  $\omega \in \Omega$  which are not rigidly periodic limit points.

### 3.2. Topological Considerations

Despite the fact that  $\Omega \setminus \Lambda \neq \emptyset$ , we prove that  $\Lambda$ , and hence  $\Pi$ , is a dense subset of  $\Omega$ .

PROPOSITION 3.5. *Every open set  $U_v^u(\omega) \subseteq \Omega$  contains rigidly periodic limit points whose periodic representatives have arbitrarily large minimal period. Moreover rigidly periodic limit points can be found whose periodic representatives are  $(r, s)$ -periodic with  $r, s > 0$ .*

*Proof.* Given any boundary point  $\omega \in \Omega$ , any vertex  $x \in \mathcal{S}_v(\omega)$ , and any  $m \in \mathbb{N}$ , we can use the proof of Theorem 2.3 to construct a rigidly periodic apartment  $\mathcal{A}$  containing  $v^{-1}I_v^u(\omega)$  and whose minimal periodicity is greater

than  $m$ . The apartment  $\mathcal{A}' = v.\mathcal{A}$  is then a rigidly periodic apartment and the sector  $\mathcal{S}_v \subset \mathcal{A}'$  containing  $I_v^u(\omega)$  defines a rigidly periodic limit point  $\omega'$  satisfying the required conditions. ■

**COROLLARY 3.6.**  *$A$  and  $\Pi$  are dense  $\Gamma$ -invariant subsets of  $\Omega$ .*

*Proof.* The density follows immediately from Proposition 3.5. To see that  $A$  and  $\Pi$  are both  $\Gamma$ -invariant, simply note that if  $\omega \in A$  has a periodic representative  $\mathcal{S} \subset \mathcal{A}$  then  $g\mathcal{S} \subset g\mathcal{A}$  is a periodic representative of  $g\omega$  of the same type. ■

### 3.3. Measure-Theoretic Considerations

In Lemma 3.9 we show that  $v_v(\Pi) = 0$  for every vertex  $v \in \Delta$ . Before we launch into the proof of this result we establish a necessary technical result.

**LEMMA 3.7.** *Let  $\mathcal{W}$  be any wall in  $\Delta$ . Let  $\Sigma_{\mathcal{W}} \subseteq \Omega$  be the set of boundary points which have representative sectors in any apartment containing  $\mathcal{W}$ . Then*

$$v_v(\Sigma_{\mathcal{W}}) = 0$$

for all vertices  $v \in \Delta$ .

*Proof.* Since the measures  $v_u$  and  $v_v$  are mutually absolutely continuous for all vertices  $u, v \in \Delta$ , it is sufficient to show that  $v_v(\Sigma_{\mathcal{W}}) = 0$  for some vertex  $v \in \Delta$ .

Fix a vertex  $v = a_{0,0} \in \mathcal{W}$  and label  $\mathcal{W}$  with respect to this vertex as described in Section 2.3. Given any apartment  $\mathcal{A} \supset \mathcal{W}$ , a labelling of  $\mathcal{A}$  can be found which is compatible with this labelling of  $\mathcal{W}$ . We consider the two distinct cases that arise.

*Case 1.* The boundary points  $\omega \in \Sigma_{\mathcal{W}}$  which have representative sectors  $\mathcal{S}_v(\omega)$  with one sector wall contained in  $\mathcal{W}$ .

All such boundary points are either contained in  $\Omega_v^{a_{i,i}}$  or  $\Omega_v^{a_{-i,-i}}$  for sufficiently large  $i \in \mathbb{N}$ . Hence the set of such boundary points has trivial measure.

*Case 2.* The boundary points  $\omega \in \Sigma_{\mathcal{W}}$  which have representative sectors  $\mathcal{S}_v(\omega)$  neither of whose sector walls is contained in  $\mathcal{W}$ .

Geometrically we are in the situation depicted in Fig. 12.

The convex hull of  $\mathcal{S} = \mathcal{S}_v(\omega)$  and  $\mathcal{W}$  in any apartment containing them both determines a root  $\mathcal{R}$  which is unique in the sense that it is independent of the particular apartment chosen. In  $\mathcal{R}$ , denote by  $\mathcal{S}_1 = \mathcal{S}_v(\omega_1)$  the unique sector based at  $v$  such that  $a_{i,i} \in \mathcal{S}_1$  for all  $i \in \mathbb{N}$ . The boundary

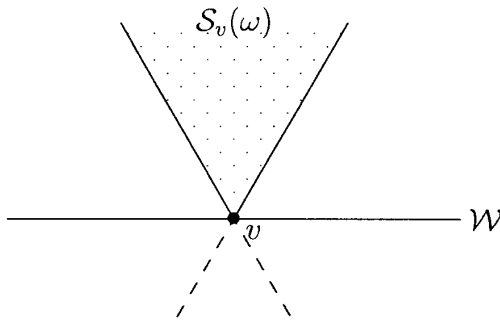


FIG. 12. Geometric interpretation of second case.

point  $\omega_1$  determined by  $\mathcal{S}_1$  satisfies the conditions for the first case considered and hence the set of all boundary points obtained in this manner belong to a set of measure zero.

There is a bijective map  $r_\Omega: \omega \mapsto \omega_1$  which is implemented in any apartment  $\mathcal{A}$  containing  $\mathcal{R}$  via a reflection in the wall separating  $\mathcal{S}$  from  $\mathcal{S}_1$  (see Fig. 13).

We note that

$$v_v(\Omega_v^u) = v_v(\Omega_v^{r(u)})$$

for all vertices  $u \in \mathcal{A}$  since  $r$  is a distance-preserving transformation of  $\mathcal{A}$ . Hence  $r_\Omega$  is a measure-preserving map.

Since the set of all boundary points covered by the first case has trivial measure, the same must therefore be true of the set of boundary points covered by the second case. ■

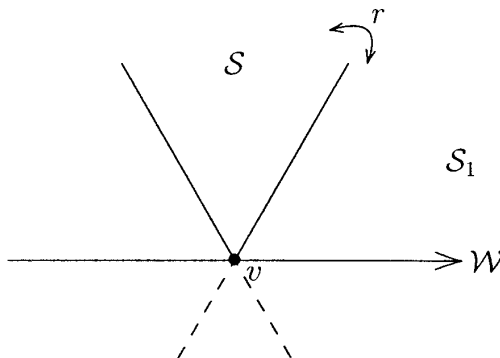


FIG. 13. Implementation of correspondence in  $\mathcal{A}$ .

We note that there are only a countable number of periodic walls in  $\mathcal{A}$ . This is because  $\Gamma$  is countable so that we have a countable number of choices for the base vertex  $a_{0,0} \in \mathcal{W}$  and, assuming  $a_{0,0} = e$ , there is a countable number of choices for the elements  $g \in \Gamma$  such that  $g\mathcal{W} = \mathcal{W}$ . We can therefore deduce the following corollary of Lemma 3.7.

**COROLLARY 3.8.** *Let  $\Sigma \subseteq \Omega$  be the set of boundary points which have representative sectors in an apartment containing a periodic wall. Then*

$$v_v(\Sigma) = 0$$

for all vertices  $v \in \mathcal{A}$ .

We are now in a position to prove the following useful result.

**LEMMA 3.9.** *Given any vertex  $v \in \mathcal{A}$ ,  $v_v(\Pi) = 0$ .*

*Proof.* Since  $\mathcal{A}$  is countable and points have measure zero,  $v_v(\mathcal{A}) = 0$  for all vertices  $v \in \mathcal{A}$ .

Now consider the set  $\Pi \setminus \mathcal{A}$ . This consists of periodic limit points whose periodic representatives are not rigidly periodic. By Lemma 2.11 any  $\omega \in \Pi \setminus \mathcal{A}$  has a periodic representative  $\mathcal{S}(\omega)$  which determines a periodic root  $\mathcal{R}$  which contains periodic walls. Hence every element  $\omega \in \Pi \setminus \mathcal{A}$  has a representative in an apartment containing a periodic wall. Thus  $\Pi \setminus \mathcal{A} \subseteq \Sigma$  and therefore  $v_v(\Pi \setminus \mathcal{A}) = 0$  for all vertices  $v \in \mathcal{A}$  by Corollary 3.8.

We deduce that  $v_v(\Pi) = 0$  as required. ■

Proposition 3.2 and Lemma 3.9 have some immediate consequences.

**PROPOSITION 3.10.** *The action of  $\Gamma$  on  $\Omega$  is measure-theoretically free; i.e.,*

$$v_v(\{\omega \in \Omega : g\omega = \omega\}) = 0$$

for all vertices  $v \in \mathcal{A}$  and elements  $g \in \Gamma \setminus \{e\}$ .

*Proof.* As a result of Proposition 3.2,

$$\{\omega \in \Omega : g\omega = \omega\} \subseteq \Pi$$

for any  $g \in \Gamma \setminus \{e\}$ . Hence

$$v_v(\{\omega \in \Omega : g\omega = \omega\}) \leq v_v(\Pi)$$

and the result follows from Lemma 3.9. ■

PROPOSITION 3.11. *The action of  $\Gamma$  on  $\Omega$  is ergodic.*

*Proof.* Since the measures  $\nu_v$  are mutually absolutely continuous, it is enough to prove this for the measure  $\nu = \nu_e$ . We must therefore show that any  $\Gamma$ -invariant function  $f \in L^\infty(\Omega, \nu)$  is constant a.e.

By the Lebesgue differentiation theorem (see [Z, Chapter 8, Sect. 36] and [Ru, Theorem 8.8]), we have for almost all  $\omega_0 \in \Omega$

$$\lim_{n \rightarrow \infty} \frac{1}{\nu(\Omega_e^{v_n})} \int_{\Omega_e^{v_n}} |f(\omega) - f(\omega_0)| \, d\nu(\omega) = 0 \tag{1}$$

where  $v_n \in \mathcal{S}_e(\omega) \cap V_e^{n,n}$ , so that  $\Omega_e^{v_n}$  is a contracting sequence of neighbourhoods of  $\omega_0$ .

Choose such a point  $\omega_0$  and write  $f(\omega_0) = \alpha$ . For each  $n \in \mathbb{N}$ , write  $v_n = g_n z_n$  where  $g_n \in V_e^{n-1, n-1}$  and  $z_n \in V_e^{1,1}$  (see Fig. 14).

Note that  $\Omega_{g_n}^{v_n} = \Omega_e^{v_n}$ . Now

$$\begin{aligned} \frac{1}{\nu(\Omega_e^{z_n})} \int_{\Omega_e^{z_n}} |f(\omega) - \alpha| \, d\nu(\omega) &= \frac{1}{\nu(\Omega_e^{z_n})} \int_{g_n \Omega_e^{z_n}} |f(g_n^{-1}\omega) - \alpha| \, d\nu(g_n^{-1}\omega) \\ &= \frac{1}{\nu(\Omega_e^{z_n})} \int_{\Omega_e^{v_n}} |f(\omega) - \alpha| \, d\nu_{g_n}(\omega) \end{aligned} \tag{2}$$

since  $g_n \Omega_e^{z_n} = \Omega_{g_n}^{g_n z_n} = \Omega_{g_n}^{v_n} = \Omega_e^{v_n}$  and  $f$  is  $\Gamma$ -invariant.

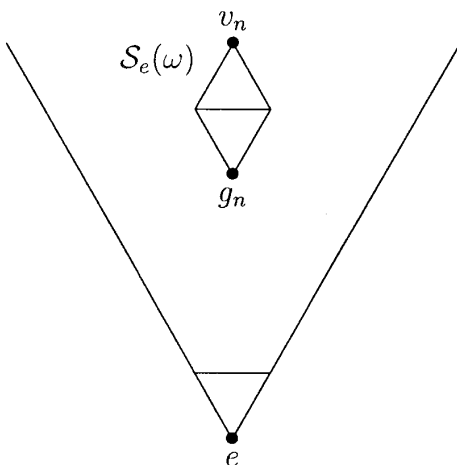


FIG. 14. Relative positions of  $v_n$  and  $g_n$ .

Furthermore, it follows from [CMS, Lemma 2.5] that, for sufficiently large  $n \in \mathbb{N}$ , the Radon–Nikodym derivative  $dv_{g_n}/dv$  is constant almost everywhere on  $\Omega_e^{v_n}$ . Moreover

$$\frac{dv_{g_n}}{dv}(\omega) = \frac{v_{g_n}(\Omega_e^{v_n})}{v(\Omega_e^{v_n})} = \frac{v(g_n^{-1}\Omega_e^{v_n})}{v(\Omega_e^{v_n})} = \frac{v(\Omega_e^{z_n})}{v(\Omega_e^{v_n})}.$$

The expression labelled (2) therefore becomes

$$\frac{1}{v(\Omega_e^{v_n})} \int_{\Omega_e^{v_n}} |f(\omega) - \alpha| dv(\omega) \tag{3}$$

Therefore, by (1) and (2),

$$\frac{1}{v(\Omega_e^{z_n})} \int_{\Omega_e^{z_n}} |f(\omega) - \alpha| dv(\omega) \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Now  $z_n$  lies in the finite set  $V_e^{1,1}$  for all  $n \in \mathbb{N}$ , so we can choose a subsequence  $n_k$  such that  $z_{n_k} = z$  for all  $k$ . Then

$$\int_{\Omega_e^z} |f(\omega) - \alpha| dv(\omega) = 0.$$

Therefore  $f(\omega) = \alpha$  for almost all  $\omega \in \Omega_e^z$ .

It follows from [RS, Lemma 3.1] that for each possible base chamber  $C = \{e, a^{-1}, b\}$  there exists an element  $g \in \Gamma$  such that  $g\Omega_e^C \subseteq \Omega_e^z$ . Hence  $f(\omega) = \alpha$  a.e.  $(v)$  on  $g\Omega_e^C$ . By the  $\Gamma$ -invariance of  $f$ ,  $f(\omega) = \alpha$  a.e.  $(v_g)$  on  $\Omega_e^C$ . Since the measures  $v, v_g$  are mutually absolutely continuous,  $f(\omega) = \alpha$  a.e.  $(v)$  on each set  $\Omega_e^C$  and hence on all of  $\Omega$ . ■

*Remark 3.12.* The above proof is based on that for the case of the free group which is contained in [PS, Proposition 3.9].

**COROLLARY 3.13.** *The von Neumann algebra  $L^\infty(\Omega) \rtimes \Gamma$  is a factor.*

*Proof.* This follows directly from Proposition 3.10, Proposition 3.11 and standard results in the theory of von Neumann algebras (see [Su, Proposition 4.1.15] for example). ■

**PROPOSITION 3.14.** *The action of  $\Gamma$  on  $\Omega$  is amenable.*

*Proof.* It is sufficient, by [A, Théorème 3.3b)], to show that there exists a sequence  $\{f_i\}_{i \in \mathbb{N}}$  of real-valued functions in  $L^\infty(\Gamma \times \Omega)$  such that

- (1)  $\sum_{g \in \Gamma} |f_i(g, \omega)|^2 = 1$  for all  $\omega \in \Omega$  and  $i \in \mathbb{N}$
- (2)  $\lim_i \sum_{g \in \Gamma} \overline{f_i(g, \omega)} f_i(h^{-1}g, h^{-1}\omega) = 1$  ultraweakly in  $L^\infty(\Omega)$  for each  $h \in \Gamma$ .

For each  $\omega \in \Omega$ , let  $f_i(\cdot, \omega)$  be the characteristic function of

$$\{g \in S_e(\omega) : |g| \leq i - 1\},$$

normalized so that the first condition holds. Thus

$$f_i(g, \omega) = \begin{cases} \left(\frac{i(i+1)}{2}\right)^{-1/2} & \text{if } g \in S_e(\omega) \text{ and } |g| \leq i - 1, \\ 0 & \text{otherwise} \end{cases}$$

and, for each  $\omega \in \Omega$ , there are exactly  $(i(i+1)/2)$  elements  $g \in \Gamma$  for which  $f_i(g, \omega) \neq 0$ . It was proved in [RS, Proposition 4.2.1] that this sequence of functions satisfies a stronger version of these conditions, since we actually have uniform convergence in the second condition. ■

**COROLLARY 3.15.** *The von Neumann algebra  $L^\infty(\Omega) \rtimes \Gamma$  is a hyperfinite factor.*

#### 4. SOME ALGEBRAS $L^\infty(\Omega) \rtimes G$

We refer to [Su, Definition 4.1.2] for the definition of the crossed product von Neumann algebra  $L^\infty(\Omega, \nu) \rtimes G$  associated with the action of a group  $G$  on a measure space  $(\Omega, \nu)$ .

In this section we investigate some von Neumann algebras which arise in this manner where  $(\Omega, \nu)$  is the measure space described in Section 1.5, paying particular attention to  $L^\infty(\Omega) \rtimes \Gamma$ . We begin by recalling some classical definitions.

**DEFINITION 4.1.** Given a group  $\Gamma$  acting on a measure space  $\Omega$ , we define the *full group*,  $[\Gamma]$ , of  $\Gamma$  by

$$[\Gamma] = \{T \in \text{Aut}(\Omega) : T\omega \in \Gamma\omega \text{ for almost every } \omega \in \Omega\}.$$

The set  $[\Gamma]_0$  of measure preserving maps in  $[\Gamma]$  is then given by

$$[\Gamma]_0 = \{T \in [\Gamma] : T \circ \nu = \nu\}$$

**DEFINITION 4.2.** Let  $G$  be a countable group of automorphisms of the measure space  $(\Omega, \nu)$ . Following Krieger, define the *ratio set*  $r(G)$  to be



the subset of  $[0, \infty)$  such that if  $\lambda \geq 0$  then  $\lambda \in r(G)$  if and only if for every  $\varepsilon > 0$  and Borel set  $\mathcal{E}$  with  $\nu(\mathcal{E}) > 0$ , there exists a  $g \in G$  and a Borel set  $\mathcal{F}$  such that  $\nu(\mathcal{F}) > 0$ ,  $\mathcal{F} \cup g\mathcal{F} \subseteq \mathcal{E}$  and

$$\left| \frac{d\nu \circ g}{d\nu}(\omega) - \lambda \right| < \varepsilon$$

for all  $\omega \in \mathcal{F}$ .

*Remark 4.3.* The ratio set  $r(G)$  depends only on the quasi-equivalence class of the measure  $\nu$ , see [HO, Section I-3, Lemma 14]. It also depends only on the full group in the sense that

$$[H] = [G] \Rightarrow r(H) = r(G).$$

Let  $\Delta$  be an arbitrary triangle building of order  $q$  with base vertex  $e$  and write  $\nu = \nu_e$ .

**PROPOSITION 4.4.** *Let  $G$  be a countable subgroup of  $\text{Aut}_{\text{tr}}(\Delta) \subseteq \text{Aut}(\Omega)$ . Suppose there exist an element  $g \in G$  such that  $d(ge, e) = 1$  and a subgroup  $K$  of  $[G]_0$  whose action on  $\Omega$  is ergodic. Then*

$$r(G) = \{q^{2n} : n \in \mathbb{Z}\} \cup \{0\}.$$

*Proof.* By Remark 4.3, it is sufficient to prove the statement for some group  $H$  such that  $[H] = [G]$ . In particular, since  $[G] = [\langle G, K \rangle]$  for any subgroup  $K$  of  $[G]_0$ , we may assume without loss of generality that  $K \leq G$ .

By [CMS, Lemma 2.5], for each  $g \in G$ ,  $\omega \in \Omega$  we have

$$\frac{d\nu \circ g}{d\nu}(\omega) \in \{q^{2n} : n \in \mathbb{Z}\} \cup \{0\}.$$

Since  $G$  acts ergodically on  $\Omega$ ,  $r(G) \setminus \{0\}$  is a group. It is therefore enough to show that  $q^2 \in r(G)$ . Write  $x = ge$  and note that  $\nu_x = \nu \circ g^{-1}$ . By [CMS, Lemmas 2.2 and 2.5] we have

$$\frac{d\nu_x}{d\nu}(\omega) = q^2, \quad \text{for all } \omega \in \Omega_e^x. \quad (4)$$

Let  $\mathcal{E} \subseteq \Omega$  be a Borel set with  $\nu(\mathcal{E}) > 0$ . By the ergodicity of  $K$ , there exist  $k_1, k_2 \in K$  such that the set

$$\mathcal{F} = \{\omega \in \mathcal{E} : k_1\omega \in \Omega_e^x \text{ and } k_2g^{-1}k_1\omega \in \mathcal{E}\}$$

has positive measure.

Finally, let  $t = k_2 g^{-1} k_1 \in G$ . By construction,  $\mathcal{F} \cup t\mathcal{F} \subseteq \mathcal{E}$ . Moreover, since  $K$  is measure-preserving,

$$\frac{dv \circ t}{dv}(\omega) = \frac{dv \circ g^{-1}}{dv}(k_1 \omega) = \frac{dv_x}{dv}(k_1 \omega) = q^2 \quad \text{for all } \omega \in \mathcal{F}$$

by (4), since  $k_1 \in \Omega_e^x$ . This proves  $q^2 \in r(G)$ , as required. ■

By definition (e.g., [HO, Section I-3]), the conclusion of Proposition 4.4 says that the action of  $G$  is of type III<sub>1/q<sup>2</sup></sub>.

**COROLLARY 4.5.** *If, in addition to the hypotheses for Proposition 4.4, the action of  $G$  is free, then  $L^\infty(\Omega) \rtimes G$  is a factor of type III<sub>1/q<sup>2</sup></sub>.*

*Proof.* Having determined the ratio set, this is immediate from [C, Corollaire 3.3.4]. ■

Thus, if we can find a countable subgroup  $K \leq [G]_0$  whose action on  $\Omega$  is ergodic we will have shown that  $L^\infty(\Omega) \rtimes G$  is a factor of type III<sub>1/q<sup>2</sup></sub>. To this end, we prove the following sufficiency condition for ergodicity.

**LEMMA 4.6.** *Let  $K$  be group which acts on  $\Omega$ . If  $K$  acts transitively on the collection of sets  $\{\Omega_e^x : x \in V_e^{m,n}\}$  for every pair  $(m, n) \in (\mathbb{N} \times \mathbb{N})$ , then  $K$  acts ergodically on  $\Omega$ .*

*Proof.* Suppose now that  $X_0 \subseteq \Omega$  is a Borel set which is invariant under  $K$  and such that  $v(X_0) > 0$ . We show  $v(\Omega \setminus X_0) = 0$ , thus establishing the ergodicity of the action.

Define a new measure  $\mu$  by  $\mu(X) = v(X \cap X_0)$  for each Borel set  $X \subseteq \Omega$ . Now, for each  $g \in K$ ,

$$\begin{aligned} \mu(gX) &= v(gX \cap X_0) = v(X \cap g^{-1}X_0) \\ &\leq v(X \cap X_0) + v(X \cap (g^{-1}X_0 \setminus X_0)) \\ &= v(X \cap X_0) \\ &= \mu(X), \end{aligned}$$

and therefore  $\mu$  is  $K$ -invariant.

For each  $u, v \in V_e^{m,n}$  there exists a  $g \in K$  such that  $g\Omega_e^u = \Omega_e^v$  by transitivity. Thus  $\mu(\Omega_e^u) = \mu(\Omega_e^v)$ . Since  $\Omega$  is the union of  $N_{m,n}$  disjoint sets  $\Omega_e^u$ ,  $u \in V_e^{m,n}$  each of which has equal measure we deduce that

$$\mu(\Omega_e^u) = \frac{c}{N_{m,n}}, \quad \text{for each } u \in V_e^{m,n},$$

where  $c = \mu(X_0) = v(X_0) > 0$ . Thus  $\mu(\Omega_e^u) = cv(\Omega_e^u)$  for every vertex  $u \in \mathcal{A}$ .

Since the sets  $\Omega_e^u$  generate the Borel  $\sigma$ -algebra, we deduce that  $\mu(X) = c\nu(X)$  for each Borel set  $X$ . Therefore

$$\begin{aligned} \nu(\Omega \setminus X_0) &= c^{-1}\mu(\Omega \setminus X_0) \\ &= c^{-1}\nu((\Omega \setminus X_0) \cap X_0) = 0, \end{aligned}$$

thus proving ergodicity. ■

The following result shows that we can assume that the group  $K \leq \text{Aut}(\Omega)$  is countable without any loss of generality.

LEMMA 4.7. *Assume that  $K \leq \text{Aut}(\Omega)$  acts transitively on the collection of sets*

$$\{\Omega_e^x : x \in V_e^{m,n}\}$$

for every pair  $(m, n) \in \mathbb{N} \times \mathbb{N}$ . Then there is a countable subgroup  $K_0$  of  $K$  which also acts transitively on the collection of sets  $\{\Omega_e^x : x \in V_e^{m,n}\}$  for every pair  $(m, n) \in \mathbb{N} \times \mathbb{N}$ .

*Proof.* For each pair  $v, w \in V_e^{m,n}$ , there exists an element  $k \in K$  such that  $k\Omega_e^u = \Omega_e^v$ . Choose one such element  $k \in K$  and label it  $k_{v,w}$ . Since  $V_e^{m,n}$  is finite, there are a finite number of elements  $k_{v,w} \in K$  for each  $V_e^{m,n}$ . There are countably many sets  $V_e^{m,n}$ , so the set  $\{k_{v,w} : v, w \in V_e^{m,n}, m, n \geq 0\}$  is countable. Hence the group

$$K_0 = \langle k_{v,w} : v, w \in V_e^{m,n}, m, n \geq 0 \rangle \leq K$$

is countable and satisfies the required condition. ■

#### 4.1. The Algebra $L^\infty(\Omega) \rtimes \Gamma$

We aim to construct a subgroup  $K \leq [\Gamma]_0$  which acts ergodically on  $\Omega$  and use Proposition 4.4 and Lemma 4.6 to prove that  $L^\infty(\Omega) \rtimes \Gamma$  is a factor of type III $_{1/q^2}$ . We begin with a technical remark.

LEMMA 4.8. *Given a fixed vertex  $x \in \Delta_{\mathcal{F}}$  and a fixed chamber  $C$  with  $x \in C$ , there are precisely  $q^3$  chambers  $D$  with the property that  $x \in D$  and for every  $\omega \in \Omega_x^D$ ,  $S_x(\omega)$  is opposite  $S_x(\varpi)$  for every  $\varpi \in \Omega_x^C$ .*

*Proof.* Since the neighbours of any vertex can be identified with the projective plane of order  $q$  introduced in Section 1.3 we may use properties of projective planes to prove this result.

The chambers incident on  $x$  correspond to point line pairs  $\{p, l\}$ . Suppose  $C$  corresponds to  $\{p_1, l_1\}$ . Then the chambers  $D$  will correspond to

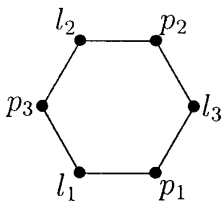


FIG. 15. Incidence diagram for point line pairs  $\{p_2, l_2\}$  corresponding to chambers  $D$ .

point line pairs  $\{p_2, l_2\}$  for which there exists an incidence diagram of the type shown in Fig. 15 for some point  $p_3$  and some line  $l_3$ .

Having fixed the pair  $\{l_1, p_1\}$  we can choose any point  $p_2 \notin l_1$ , so there are  $(q^2 + q + 1) - (q + 1) = q^2$  choices for  $p_2$ . Having chosen  $p_2$ , the line  $l_3$  is then uniquely determined.

The number of possible choices for the line  $l_2$  is then determined by the number of possible choices for the point  $p_3$ . The only restrictions on  $p_3$  are that  $p_3 \in l_1$  but  $p_3 \neq p_1$ . So there are  $(q + 1) - 1 = q$  choices for  $p_3$ , and hence for  $l_2$ .

Hence there are  $q^2q = q^3$  possible pairs  $\{p_2, l_2\}$  satisfying the necessary conditions. ■

We can now construct the ergodic subgroup of  $[\Gamma]_0$  provided  $q \geq 3$ .

PROPOSITION 4.9. *If  $q \geq 3$  there is a countable ergodic group  $K \leq \text{Aut}(\Omega)$  such that  $K \leq [\Gamma]_0$ .*

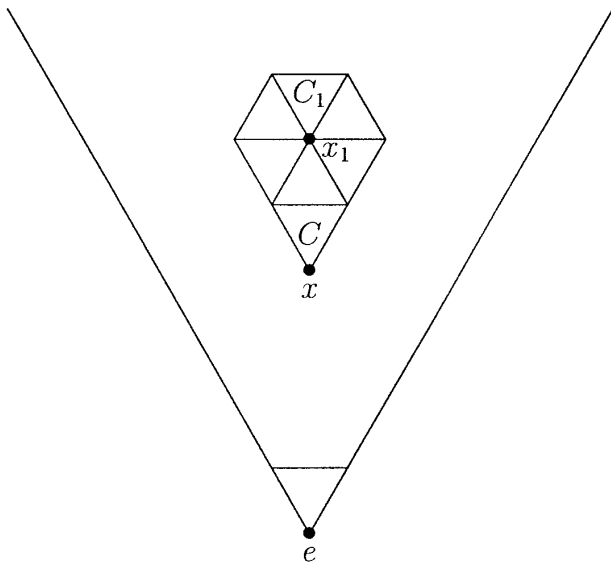


FIG. 16. Relative positions of  $x, x_1, C,$  and  $C_1$ .

*Proof.* Let  $x, y \in V_e^{m,n}$ . We construct a measure preserving automorphism  $k_{x,y}$  of  $\Omega$  such that

- (1)  $k_{x,y}$  is almost everywhere a bijection from  $\Omega^x$  onto  $\Omega^y$ ,
- (2)  $k_{x,y}$  is the identity on  $\Omega \setminus (\Omega^x \cup \Omega^y)$ .

It then follows from Lemma 4.6 that the group

$$K = \langle k_{x,y} : \{x, y\} \subseteq V_e^{m,n}, m, n \in \mathbb{N} \rangle$$

acts ergodically on  $\Omega$  and the construction will show explicitly that  $K \leq [\Gamma]_0$ .

Recall from Section 1.5 that the Borel  $\sigma$ -algebra is generated by sets of the form  $\Omega_v^u$  and that such a set is the disjoint union of open sets of the form  $\Omega_C^v$  for some chamber  $C$  with  $v \in C$ . It is therefore enough to show that, for every  $x, y \in V_e^{m,n}$  and chambers  $C$  and  $D$  satisfying  $x \in C, y \in D, \Omega_x^C = \Omega_e^C$  and  $\Omega_y^D = \Omega_e^D$ , there exists an almost everywhere bijection

$$k: \Omega_x^C \rightarrow \Omega_y^D$$

which is measure preserving and is pointwise approximable by the action of  $\Gamma$  almost everywhere on  $\Omega_x^C$ .

As depicted in Fig. 16 the set  $\Omega_x^C$  is a disjoint union of sets of the form  $\Omega_{x_1}^{C_1}$  where  $x_1 \in V_e^{m+1,n+1}$  and  $\Omega_{x_1}^{C_1} = \Omega_e^{C_1}$ . Similarly the set  $\Omega_y^D$  is a disjoint union of sets of the form  $\Omega_{y_1}^{D_1}$  where  $y_1 \in V_e^{m+1,n+1}$  and  $\Omega_{y_1}^{D_1} = \Omega_e^{D_1}$ .

Fix two such vertices  $x_1, y_1$ . By Lemma 4.8 there are  $q^3$  possible choices for each of the chambers  $C_1$  and  $D_1$ .

Now  $x_1^{-1}C_1$  and  $y_1^{-1}D_1$  are each one of the  $\alpha = (q+1)(q^2+q+1)$  chambers based at  $e$ . Therefore, since  $2q^3 > \alpha$  for  $q \geq 3$ , we can choose chambers  $C_1$  and  $D_1$  such that  $x_1^{-1}C_1 = y_1^{-1}D_1$ , i.e., such that  $y_1x_1^{-1}C_1 = D_1$ . Define  $k$  from  $\Omega_{x_1}^{C_1}$  onto  $\Omega_{y_1}^{D_1}$  by

$$k\omega = y_1x_1^{-1}\omega \quad \text{for } \omega \in \Omega_{x_1}^{C_1}.$$

Thus the action of  $k$  is measure preserving and pointwise approximable by  $\Gamma$  on  $\Omega_{x_1}^{C_1}$ . Therefore  $k$  remains undefined on a proportion  $(\alpha-1)/\alpha$  of  $\Omega_x^C$ .

Now repeat the process on each of the pairs of sets  $\Omega_{x_1}^{C_1}, \Omega_{y_1}^{D_1}$  for which  $k$  has not been defined. As before,  $k$  can be defined everywhere except on a proportion  $(\alpha-1)/\alpha$  of each such set, and  $k$  can therefore be defined everywhere except on a proportion  $((\alpha-1)/\alpha)^2$  of the original set  $\Omega_x^C$ .

Continuing in this manner we find that at the  $n$ th step,  $k$  has been defined everywhere except on a proportion  $((\alpha - 1)/\alpha)^n$  of  $\Omega_x^C$ . Since

$$\left(\frac{\alpha - 1}{\alpha}\right)^n \rightarrow 0 \quad \text{as } n \rightarrow \infty,$$

$k$  is defined almost everywhere on  $\Omega_x^C$  and satisfies the required properties. ■

The above considerations lead us to the following conclusion.

**THEOREM 4.10.** *The von Neumann algebra  $L^\infty(\Omega) \rtimes \Gamma$  is a hyperfinite factor.*

*Moreover, if  $q \geq 3$  the action of  $\Gamma$  on  $\Omega$  is of type III<sub>1/q<sup>2</sup></sub> and so  $L^\infty(\Omega) \rtimes \Gamma$  is the hyperfinite factor of type III<sub>1/q<sup>2</sup></sub>.*

*Proof.* The von Neumann algebra  $L^\infty(\Omega) \rtimes \Gamma$  is a hyperfinite factor by Corollary 3.13 and Corollary 3.15.

If  $q \geq 3$ , then Propositions 4.6 and 4.9 prove that the factor is of type III<sub>1/q<sup>2</sup></sub>. ■

*Remark 4.11.* We believe that the result is also true when  $q = 2$ , but the proof of Proposition 4.9 in that case appears to be harder.

#### 4.2. Algebras from Classical Groups

We now restrict our attention to the triangle buildings associated to certain linear groups. Henceforth, let  $\mathcal{G} = \text{PGL}(3, \mathbb{F})$  for  $\mathbb{F}$  a local field with a discrete valuation and let  $\mathcal{K} = \text{PGL}(3, \mathcal{O})$  where  $\mathcal{O}$  is the valuation ring of  $\mathbb{F}$ . Let  $\Delta$  be the triangle building associated to  $\mathcal{G}$  [R, Chapter 9] and  $\Omega$  its boundary. Then  $\mathcal{K}$  satisfies the conditions of Lemma 4.6 by the remark following [CMS, Proposition 4.2].

**PROPOSITION 4.12.**  *$\mathcal{G}$  acts freely on  $\Omega$ .*

*Proof.* Denote by  $\mathcal{P} \leq \mathcal{G}$  the group of upper triangular matrices in  $\mathcal{G}$ . By [Br, Proposition VI.9F],  $\Omega$  is isomorphic to  $\mathcal{G}/\mathcal{P}$  as a topological  $\mathcal{G}$ -space. Moreover,  $\nu$  corresponds to the unique quasi-invariant measure on  $\mathcal{G}/\mathcal{P}$ .

Note that if  $\mathcal{F}$  is a closed subgroup of  $\mathcal{G}$  then the quasi-invariant measure  $\mu_{\mathcal{G}/\mathcal{F}}$  on  $\mathcal{G}/\mathcal{F}$  has the property that  $\mu_{\mathcal{G}/\mathcal{F}}(Y) = 0$  if and only if  $\mu_{\mathcal{G}}(\pi^{-1}(Y)) = 0$  where  $\mu_{\mathcal{G}}$  denotes left Haar measure on  $\mathcal{G}$  and  $\pi$  denotes the quotient map  $\mathcal{G} \rightarrow \mathcal{G}/\mathcal{F}$  (see [B, VII, Section 2, Théorème 1]).

Let  $g \in \mathcal{G} \setminus \{0\}$ . We must show that  $\nu(\{\omega \in \Omega : g\omega = \omega\}) = 0$ . That is to say we must show

$$\mu_{\mathcal{G}/\mathcal{P}}\{h\mathcal{P} : h^{-1}gh \in \mathcal{P}\} = 0,$$

or, equivalently, that

$$\mu_{\mathcal{G}}\{h \in \mathcal{G} : h^{-1}gh \in \mathcal{P}\} = 0.$$

The condition  $h^{-1}gh \in \mathcal{P}$  means that any representative of  $h$  in  $\text{GL}(3, \mathbb{F})$  lies in the zero set of some nonzero polynomial  $\psi \in \mathbb{F}[X_1, \dots, X_q]$ . The zero set of  $\psi$  in  $\mathbb{F}^q$  has measure zero, relative to the usual Haar measure [B, VII, Lemme 9]. The assertion follows from the explicit expression for the Haar measure in  $\text{GL}(3, \mathbb{F})$  [B, VII, Section 1, Exemple 1]. ■

**THEOREM 4.13.** (1) *Let  $q$  be a prime power and let  $\Omega$  be the boundary of the building associated to  $\text{PGL}(3, \mathbb{F}_q((X)))$ . Then  $L^\infty(\Omega) \rtimes \text{PGL}(3, \mathbb{F}_q(X))$  is a factor of type  $\text{III}_{1/q^2}$ .*

(2) *Let  $p$  be a prime and  $\Omega$  the boundary of the building associated to  $\text{PGL}(3, \mathbb{Q}_p)$ . Let  $\mathcal{R}$  denote either  $\mathbb{Z}[1/p]$  or  $\mathbb{Q}$ . Then  $L^\infty(\Omega) \rtimes \text{PGL}(3, \mathcal{R})$  is a factor of type  $\text{III}_{1/p^2}$ .*

*Proof.* (1) We apply Proposition 4.4 with  $G = \text{PGL}(3, \mathbb{F}_q(X))$  and  $K = \text{PGL}(3, \mathbb{F}_q[X])$ . The action of  $G$  is free by Proposition 4.12. We must check that the action of  $K$  is ergodic. Let  $K' = \text{PGL}(3, \mathbb{F}_q[[X]])$ . Then  $K'$  acts transitively on each set  $V_e^{m,n}$  by [CMS, remark following Proposition 4.2]. Also  $K$  is dense in  $K'$  and the action of  $K'$  on  $\Omega$  is continuous. It follows from the definition of the topology on  $\Omega$  that  $K$  also acts transitively on each  $V_e^{m,n}$ . Thus Lemma 4.6 ensures that  $K$  acts ergodically on  $\Omega$ . Since the action of  $G$  is free and ergodic the crossed product  $L^\infty(\Omega) \rtimes G$  is therefore a factor.

In order to verify the remaining hypothesis of Proposition 4.4, suppose that the base vertex  $e$  of  $\Delta$  is represented by the lattice class of  $\langle e_1, e_2, e_3 \rangle$ . Then the lattice class of  $\langle Xe_1, e_2, e_3 \rangle$  represents a neighbouring vertex, so the element  $g \in G$  represented by the matrix

$$\begin{pmatrix} X & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

satisfies  $d(ge, e) = 1$ .

The result now follows from Proposition 4.4.

(2) The proof is analogous to that given for the first statement but with  $K = \text{PGL}(3, \mathbb{Z})$  and  $G = \text{PGL}(3, \mathcal{R})$ . ■

*Remarks 4.14.* (1) We note that since the action of  $\text{PGL}(3, \mathbb{Z})$  in Theorem 4.13 is measure-preserving, it follows that  $L^\infty(\Omega) \rtimes \text{PGL}(3, \mathbb{Z})$  is a factor of type  $\text{II}_1$ .

(2) The real analogue of Theorem 4.13 is known. Take  $G = \mathrm{PGL}(n, \mathbb{R})$  with  $n > 1$  and  $\Omega = G/P$  where  $P$  is the group of upper triangular matrices in  $G$ . It then follows from a result of D. Sullivan and R. Zimmer (see [S]) that  $L^\infty(\Omega) \rtimes \mathrm{PGL}(n, \mathbb{Q})$  is a factor of type III<sub>1</sub>.

(3) The building associated to  $\mathrm{PGL}(2, \mathbb{F}_q((X)))$  is a homogeneous tree of degree  $q + 1$ . In this case the corresponding factor is of type III<sub>1/q</sub>. The building associated to  $\mathrm{PGL}(n, \mathbb{F}_q((X)))$  is a simplicial complex of rank  $n - 1$  and degree  $q + 1$ . Preliminary investigations indicate that a similar construction will yield a factor of type III<sub>1/q</sub> if  $n$  is odd and III<sub>1/q<sup>2</sup></sub> if  $n$  is even.

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