Théorie des Groupes/*Group Theory*

AFFINE KAC-MOODY GROUPS OF TYPES II AND III

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Abstract. We announce a theorem which states that, over certain fields, affine Kac-Moody groups of types II and III arise as the fixed point subgroups under particular automorphisms of affine Kac-Moody groups obtained from simply-laced extended Cartan matrices (and hence of type I) of higher rank. Thus our result extends a theorem on Kac-Moody algebras to corresponding groups. A detailed proof of this result will appear in the *Journal of Algebra*.

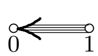
GROUPES DE KAC-MOODY AFFINES DE TYPES II ET III

Résumé. On donne un théorème qui étend un résultat déjà connu sur les algèbres de Kac-Moody aux groupes correspondants. Notre résultat affirme que, sur certains corps, les groupes de Kac-Moody affines de type II et de type III peuvent être réalisés comme les sous-groupes des points fixés par des automorphismes particuliers dans des groupes de Kac-Moody affines du type I de rang supérieur, plus précisément dans des groupes déduits de matrices de Cartan étendues simplement lacées. La démonstration détaillée de ce résultat sera publiée au *Journal of Algebra*.

Version Française Abrégée

Il existe des automorphismes diagonaux des groupes affines de Kac-Moody de type I qui ne sont pas intérieurs même si le corps est algébriquement clos (voir [1]); ceci n'est pas le cas pour les groupes de Chevalley. Dans le présent travail, on considère les sous-groupes des groupes de types $\tilde{A}_l$, $\tilde{D}_l$, et $\tilde{E}_6$, formés des points fixés par des automorphismes produits d'un automorphisme de graphe et d'un automorphisme diagonal. Sur certains corps de base, par exemple sur le corps des nombres complexes, les groupes obtenus par ce procédé sont isomorphes aux groupes de Kac-Moody affines de type II et III; cela fournit une description explicite de ces groupes.

On note ${}^tA$ la matrice déduite de A par transposition et ${}^*\tilde{A}_1$ et ${}^*\tilde{C}_l$ les matrices dont les diagrammes de Dynkin sont



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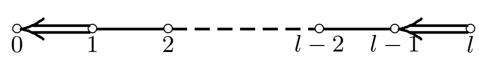

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respectivement. Les foncteurs en groupes de type minimal adjoint, adjoint et simplement connexe qui correspondent à la matrice de Cartan A seront notés $\mathcal{G}^{\mathrm{A}}_{\mathrm{m}}$, $\mathcal{G}^{\mathrm{A}}_{\mathrm{ad}}$, et $\mathcal{G}^{\mathrm{A}}_{\mathrm{sc}}$ respectivement. Etant donnés $\gamma, \tau \in \operatorname{Aut} \mathcal{G}^{\mathrm{A}}_{\mathrm{m}}$, l'on pose $\mathcal{G}^{\gamma,\tau}_{\mathrm{m}} = \{x \in \mathcal{G}^{\mathrm{A}}_{\mathrm{m}} : \gamma(x) = \tau(x)\}$. On définit $\mathcal{G}^{\hat{\gamma},\hat{\tau}}_{\mathrm{ad}}$ et $\mathcal{G}^{\gamma',\tau'}_{\mathrm{sc}}$ de manière analogue, pour $\hat{\gamma}, \hat{\tau} \in \operatorname{Aut} \mathcal{G}^{\mathrm{A}}_{\mathrm{ad}}$, et $\gamma', \tau' \in \operatorname{Aut} \mathcal{G}^{\mathrm{A}}_{\mathrm{sc}}$. Notre résultat s'énonce:

Théorème . *Soit B une matrice de Cartan affine de type II ou III. Associons-lui la matrice A et l'entier k définis par la table 1 ci-dessous (A est une matrice de Cartan affine simplement lacée de rang supérieur au rang de B, et k est égal à 2 ou à 3). Il existe*

- *des automorphismes γ, $\hat{\gamma}$, γ' et τ, $\hat{\tau}$, τ', d'ordre k, des foncteurs $\mathcal{G}^{\mathrm{A}}_{\mathrm{m}}$, $\mathcal{G}^{\mathrm{A}}_{\mathrm{ad}}$, et $\mathcal{G}^{\mathrm{A}}_{\mathrm{sc}}$, et*
- *des isomorphismes de foncteurs en groupes*

$$\begin{array}{rcccl} \Psi_m & : & \mathcal{G}^{\mathrm{B}}_{\mathrm{m}} & \to & \mathcal{G}^{\gamma,\tau}_{\mathrm{m}}, \\ \Psi_{ad} & : & \mathcal{G}^{\mathrm{B}}_{\mathrm{ad}} & \to & \mathcal{G}^{\hat{\gamma},\hat{\tau}}_{\mathrm{ad}}, \\ \Psi_{sc} & : & \mathcal{G}^{\mathrm{B}}_{\mathrm{sc}} & \to & \mathcal{G}^{\gamma',\tau'}_{\mathrm{sc}}, \end{array}$$

où les foncteurs en groupes sont définis sur la catégorie des corps $\mathbb{K}$ tels que $\operatorname{char} \mathbb{K} \neq 2$ et qui satisfont en outre

$$\text{si } B = {}^*\tilde{A}_1 \text{ ou } {}^*\tilde{C}_l, \text{ à la condition: } \sqrt{2} \in \mathbb{K}, \text{ et}$$
$$\text{si } B = {}^t\tilde{G}_2, \text{ aux conditions: } \operatorname{char} \mathbb{K} \neq 3 \text{ et } \mathbb{K} \text{ contient une racine cubique primitive de l'unité.}$$

Les automorphismes γ, $\hat{\gamma}$, γ' sont des automorphismes de graphe qui fixent le sommet d'indice zéro du diagramme de Dynkin de A; les automorphismes τ, $\hat{\tau}$, τ' sont des automorphismes diagonaux. Ces automorphismes sont décrits de manière explicite dans [6] et [7].

1. Introduction

In [8],using his presentation of Chevalley groups, R. Steinberg gives a uniform construction of certain fixed point subgroups of some Chevalley groups under particular automorphisms. When the automorphism has non-trivial field and graph automorphism constituents the fixed point subgroups give rise to finite simple groups which are not covered by the general theory of Chevalley groups (e.g. certain projective special unitary groups).

In the case when there is a single root length (i.e. the simply-laced case), the fixed point subgroup of a graph automorphism alone yields only an imbedding of one Chevalley group in another. Furthermore, the remaining automorphisms of the Chevalley group in this case differ from those already considered only by an inner automorphism (at least when the group is over an algebraically closed field).

Kac-Moody groups over fields were shown to be the natural analogues of Chevalley groups by J. Tits in his fundamental paper [9].

In contrast to the case of Chevalley groups, there are diagonal automorphisms of affine Kac-Moody groups of type I which are not inner automorphisms even when the field is algebraically closed, ref. [1]. It is thus of interest to consider the fixed point subgroups of affine Kac-Moody groups under automorphisms which involve non-trivial graph and diagonal constituents.

We restrict our attention to affine Kac-Moody groups of types $\tilde{A}_l$, $\tilde{D}_l$, and $\tilde{E}_6$. Whenever there is also a non-trivial field automorphism constituent, the conditions of Hée's theorem on twisted groups are satisified (see [2] for a statement of Hée's theorem and [3, Théorème(4.5)] for a proof). A detailed analysis of these twisted Kac-Moody groups can be found in [5] and [7]. However, the fixed point subgroups under graph-by-diagonal automorphisms obtained from these simply-laced affine Kac-Moody groups fail to satisfy one of the conditions of Hée's theorem, namely the non-triviality of the twisted root subgroups. Over certain fields, e.g over $\mathbb{C}$, the twisted groups obtained in this manner from simply-connected, adjoint and minimal adjoint groups turn out to be isomorphic to affine Kac-Moody groups of types II and III defined by the corresponding root data, thus providing a realization of such groups. A detailed proof of this result can be found in [6] and [7].

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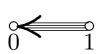
2. The Statement of the Theorem

We shall assume a certain familiarity with [2], [9] and the notation introduced therein. Throughout we shall denote by $\underline{n}$ the set of integers $\{1, \ldots, n\}$ and by $\underline{n}_0$ the set of integers $\{0, 1, \ldots, n\}$ for each integer $n \geq 1$. We denote by ${}^tA$ the matrix obtained from A by transposition and by ${}^*\tilde{A}_1$ and ${}^*\tilde{C}_l$ the matrices with Dynkin diagrams



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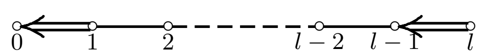


respectively. The minimal adjoint, adjoint, and simply-connected group functors corresponding to the generalized Cartan matrix A are denoted by $\mathcal{G}_{\mathrm{m}}^{\mathrm{A}}$, $\mathcal{G}_{\mathrm{ad}}^{\mathrm{A}}$, and $\mathcal{G}_{\mathrm{sc}}^{\mathrm{A}}$ respectively. For $\gamma, \tau \in \operatorname{Aut} \mathcal{G}_{\mathrm{m}}^{\mathrm{A}}$, we define $\mathcal{G}_{\mathrm{m}}^{\gamma,\tau} = \{x \in \mathcal{G}_{\mathrm{m}}^{\mathrm{A}} : \gamma(x) = \tau(x)\}$. We make similar definitions for $\mathcal{G}_{\mathrm{ad}}^{\hat{\gamma},\hat{\tau}}$ and $\mathcal{G}_{\mathrm{sc}}^{\gamma',\tau'}$. Our result can then be stated as follows.

Theorem 2.1. *Suppose B is an affine Cartan matrix of type II or III. We associate to B the matrix A and the integer k defined in table 1 (A is a simply-laced affine Cartan matrix of higher rank than B, and k is either 2 or 3). There exist*

- *automorphisms γ, $\hat{\gamma}$, γ' and τ, $\hat{\tau}$, τ', of order k, of the group functors $\mathcal{G}_{\mathrm{m}}^{\mathrm{A}}$, $\mathcal{G}_{\mathrm{ad}}^{\mathrm{A}}$, and $\mathcal{G}_{\mathrm{sc}}^{\mathrm{A}}$, and*
- *group functor isomorphisms*

$$\begin{array}{rcccl} \Psi_m & : & \mathcal{G}_{\mathrm{m}}^{\mathrm{B}} & \to & \mathcal{G}_{\mathrm{m}}^{\gamma,\tau}, \\ \Psi_{ad} & : & \mathcal{G}_{\mathrm{ad}}^{\mathrm{B}} & \to & \mathcal{G}_{\mathrm{ad}}^{\hat{\gamma},\hat{\tau}}, \\ \Psi_{sc} & : & \mathcal{G}_{\mathrm{sc}}^{\mathrm{B}} & \to & \mathcal{G}_{\mathrm{sc}}^{\gamma',\tau'}, \end{array}$$

where the group functors are defined on the category of fields $\mathbb{K}$ with $\operatorname{char}\mathbb{K} \neq 2$ and such that

$$\text{if } B = {}^*\tilde{A}_1 \text{ or } {}^*\tilde{C}_l \text{ then } \sqrt{2} \in \mathbb{K} \text{ and}$$
$$\text{if } B = {}^t\tilde{G}_2 \text{ then } \operatorname{char}\mathbb{K} \neq 3 \text{ and } \mathbb{K} \text{ contains a primitive cube root of unity.}$$

The automorphisms γ, $\hat{\gamma}$, γ' are graph automorphisms fixing the zeroth node in the Dynkin diagram of A, whereas the automorphisms τ, $\hat{\tau}$, τ' are diagonal automorphisms. All of the automorphisms are explicitly described in both [6] and [7].

B	${}^*\tilde{A}_1$	${}^*\tilde{C}_l,\ (l>1)$	${}^t\tilde{B}_l,\ (l>2)$	${}^t\tilde{C}_l,\ (l>1)$	${}^t\tilde{G}_2$	${}^t\tilde{F}_4$
(A,k)	$(\tilde{A}_2,2)$	$(\tilde{A}_{2l},2)$	$(\tilde{A}_{2l-1},2)$	$(\tilde{D}_{l+1},2)$	$(\tilde{D}_4,3)$	$(\tilde{E}_6,2)$

Table 1. Matrices A and orders k of automorphisms realizing B.

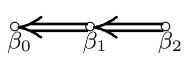
3. An Outline of the Proof

We begin by constructing certain graph and diagonal automorphisms of minimal adjoint, adjoint and simply-connected Kac-Moody groups associated to simply-laced affine matrices. We adapt Hée's theorem [3, Théorème(4.5)] so that it applies to our situation and we use it to deduce that our twisted Kac-Moody groups are endowed with the structure of a $(B\,,N)$-pair and a system of root subgroups. We show that by omitting the root orbits which are stable under the graph automorphism but whose corresponding twisted root subgroup is trivial we are left with a root system which is identified in [4] as being an affine root system of type II or III. We then prove that the groups constructed in this manner are in fact isomorphic to the corresponding groups defined via generators and relations as in [9].

The initial step in the construction of the isomorphisms is to exploit the faithful action of the minimal adjoint groups on their corresponding algebras. Denoting by $\mathfrak{g}_A(\mathbb{C})$ and $\mathfrak{g}_{\mathrm{B}}(\mathbb{C})$ the standard Kac-Moody algebras associated to the Cartan matrices A and B, we compare the actions of $\mathcal{G}_{\mathrm{m}}^{\gamma,\tau}(\mathbb{C})$ and $\mathcal{G}_{\mathrm{m}}^{\mathrm{B}}(\mathbb{C})$ on a subalgebra of $\mathfrak{g}_A(\mathbb{C})$ which we know to be isomorphic to the algebra $\mathfrak{g}_{\mathrm{B}}(\mathbb{C})$ from [4]. We use this to construct an isomorphism $\Psi_m(\mathbb{C})$ and then show that this extends to a group functor isomorphism Ψ_m. We then use our knowledge of Ψ_m to construct analogous maps Ψ_{sc} and Ψ_{ad} between the simply-connected and adjoint group functors respectively and we use group-theoretic considerations to show that these are indeed group functor isomorphisms.

4. Example

We consider the case when $B = {}^*\tilde{C}_2$, and $A = \tilde{A}_4$, i.e. the matrices with Dynkin diagrams



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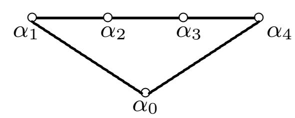


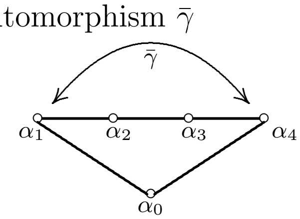
respectively, with the indicated labelling.

In [9], the simply-connected group functor $\mathcal{G}_{\mathrm{sc}}^{\mathrm{B}}$ is defined in terms of generators and relations. For each field $\mathbb{K}$, the group $\mathcal{G}_{\mathrm{sc}}^{\mathrm{B}}(\mathbb{K})$ is generated by elements $y_{\beta_j}(\mu)$, $y_{-\beta_j}(\mu)$, and $h_{\beta_j}(\xi)$ for $j \in \underline{2}_0$, $\mu \in \mathbb{K}$ and $\xi \in \mathbb{K} \setminus \{0\}$, subject to relations analogous to the Steinberg relations for Chevalley groups. Similarly we shall denote the generators of the simply-connected group $\mathcal{G}_{\mathrm{sc}}^{\mathrm{A}}(\mathbb{K})$ by $x_{\alpha_i}(\mu)$, $x_{-\alpha_i}(\mu)$, $h_{\alpha_i}(\xi)$ for $i \in \underline{4}_0$, $\mu \in \mathbb{K}$ and $\xi \in \mathbb{K} \setminus \{0\}$. We adopt Tits' convention for the choice of generators throughout.

Proofs of the following assertions regarding automorphisms of $\mathcal{G}_{\mathrm{sc}}^{\mathrm{A}}$ can be found in [1, p. 59–61] (where the results are stated and proved for $\mathbb{K} = \mathbb{C}$, but in fact extend to an arbitrary field). The map

$$\begin{array}{lcll} x_{\pm\alpha_0}(\mu) & \mapsto & x_{\pm\alpha_0}(-\mu), & \text{for } \mu \in \mathbb{K} \\ x_{\pm\alpha_i}(\mu) & \mapsto & x_{\pm\bar{\gamma}(\alpha_i)}(\mu) & \text{for } i \in \underline{4} \text{ and } \mu \in \mathbb{K} \\ h_{\alpha_i}(\xi) & \mapsto & h_{\bar{\gamma}(\alpha_i)}(\xi) & \text{for } i \in \underline{4}_0 \text{ and } \xi \in \mathbb{K} \setminus \{0\} \end{array}$$

extends to an automorphism $\gamma' : \mathcal{G}_{\mathrm{sc}}^{\mathrm{A}}(\mathbb{K}) \to \mathcal{G}_{\mathrm{sc}}^{\mathrm{A}}(\mathbb{K})$ of order 2 corresponding to the graph automorphism $\bar{\gamma}$



of the Dynkin diagram of A. Also, the map

$$\begin{array}{lll} x_{\pm\alpha_0}(\mu) & \mapsto & x_{\pm\alpha_0}(-\mu) \quad \text{for } \mu \in \mathbb{K} \\ x_{\pm\alpha_i}(\mu) & \mapsto & x_{\pm\alpha_i}(\mu) \quad \text{for } i \in \underline{4} \text{ and } \mu \in \mathbb{K} \\ h_{\alpha_i}(\xi) & \mapsto & h_{\alpha_i}(\xi) \quad \text{for } i \in \underline{4}_0 \text{ and } \xi \in \mathbb{K} \setminus \{0\} \end{array}$$

extends to a diagonal automorphism $\tau' : \mathcal{G}^{\mathrm{A}}_{\mathrm{sc}}(\mathbb{K}) \to \mathcal{G}^{\mathrm{A}}_{\mathrm{sc}}(\mathbb{K})$. Define $n_{\alpha_3} = x_{\alpha_3}(1)x_{-\alpha_3}(1)x_{\alpha_3}(1)$ and, for each $\mu \in \mathbb{K}$, let $x_{\alpha_2+\alpha_3}(\mu) = n_{\alpha_3}x_{\alpha_2}(\mu)n_{\alpha_3}^{-1}$ and $x_{-\alpha_2-\alpha_3}(\mu) = n_{\alpha_3}x_{-\alpha_2}(\mu)n_{\alpha_3}^{-1}$.

Now, in accordance with the Theorem, we suppose that $\mathbb{K}$ is a field with $\operatorname{char}\mathbb{K} \neq 2$ and $\sqrt{2} \in \mathbb{K}$. The map

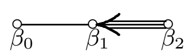
$$\begin{array}{lll} y_{\beta_0}(\mu) & \mapsto & x_{\alpha_2}(\sqrt{2}\mu)x_{\alpha_3}(\sqrt{2}\mu)x_{\alpha_2+\alpha_3}(-\mu^2) \\ y_{-\beta_0}(\mu) & \mapsto & x_{-\alpha_2}(\sqrt{2}\mu)x_{-\alpha_3}(\sqrt{2}\mu)x_{-\alpha_2-\alpha_3}(-\mu^2) \\ y_{\pm\beta_1}(\mu) & \mapsto & x_{\pm\alpha_1}(\mu)x_{\pm\alpha_4}(\mu) \\ y_{\pm\beta_2}(\mu) & \mapsto & x_{\pm\alpha_0}(\mu) \\ h_{\beta_0}(\xi) & \mapsto & h_{\alpha_2}(\xi^2)h_{\alpha_3}(\xi^2) \\ h_{\beta_1}(\xi) & \mapsto & h_{\alpha_1}(\xi)h_{\alpha_4}(\xi) \\ h_{\beta_2}(\xi) & \mapsto & h_{\alpha_0}(\xi) \end{array}$$

defined on the generators of $\mathcal{G}^{\mathrm{B}}_{\mathrm{sc}}(\mathbb{K})$ extends to an isomorphism $\Psi_{sc}(\mathbb{K}) : \mathcal{G}^{\mathrm{B}}_{\mathrm{sc}}(\mathbb{K}) \to \mathcal{G}^{\gamma',\tau'}_{\mathrm{sc}}(\mathbb{K})$.

We note that in fact $\mathcal{G}^{\gamma',\tau'}_{\mathrm{sc}}(\mathbb{K})$ is the fixed point subgroup of $\mathcal{G}^{\mathrm{A}}_{\mathrm{sc}}(\mathbb{K})$ under the automorphism σ which extends the map $x_{\pm\alpha_i}(\mu) \mapsto x_{\pm\bar\gamma(\alpha_i)}(\mu)$, $h_{\alpha_i}(\xi) \mapsto h_{\bar\gamma(\alpha_i)}(\xi)$ on the generators of $\mathcal{G}^{\mathrm{A}}_{\mathrm{sc}}(\mathbb{K})$. However, the decomposition of σ into the product of a graph and a diagonal automorphism is a feature common to all of the constructions and is therefore worth emphasizing.

5. Another Example

We now consider the case when $B = {}^t\tilde{G}_2 = (G_{jk})_{j,k\in\underline{2}_0}$, and $A = \tilde{D}_4 = (D_{il})_{i,l\in\underline{4}_0}$, i.e. the matrices with Dynkin diagrams



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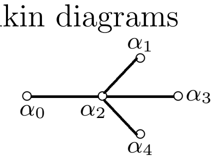
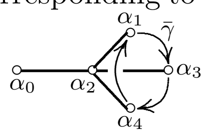


respectively, with the indicated labelling. For each field $\mathbb{K}$, the minimal adjoint group $\mathcal{G}^{\mathrm{B}}_{\mathrm{m}}(\mathbb{K})$ is generated by elements $y_{\beta_j}(\mu)$, $y_{-\beta_j}(\mu)$ and $h_{\varpi^\vee_{\beta_j}}(\xi)$ where $j \in \underline{2}_0$, $\mu \in \mathbb{K}$, $\xi \in \mathbb{K} \setminus \{0\}$ and $\{\varpi^\vee_{\beta_j}\}_{j\in\underline{2}_0}$ is a system of fundamental coweights with $\beta^\vee_j = \sum_{k\in\underline{2}_0} G_{jk}\varpi^\vee_{\beta_k}$ for $j \in \underline{2}_0$. Similarly the group $\mathcal{G}^{\mathrm{A}}_{\mathrm{m}}(\mathbb{K})$ is generated by elements $x_{\alpha_i}(\mu)$, $x_{-\alpha_i}(\mu)$ and $h_{\varpi^\vee_{\alpha_i}}(\xi)$ for $i \in \underline{4}_0$, $\mu \in \mathbb{K}$, $\xi \in \mathbb{K} \setminus \{0\}$ and fundamental coweights $\{\varpi^\vee_{\alpha_i}\}_{i\in\underline{4}_0}$ satisfying $\alpha^\vee_i = \sum_{l\in\underline{4}_0} D_{il}\varpi^\vee_{\alpha_l}$ for each $i \in \underline{4}_0$. We show that the map

$$x_{\pm\alpha_i}(\mu) \mapsto x_{\pm\bar\gamma(\alpha_i)}(\mu), \qquad h_{\varpi^\vee_{\alpha_i}}(\xi) \mapsto h_{\varpi^\vee_{\bar\gamma(\alpha_i)}}(\xi)$$

for $i \in \underline{4}_0$, $\mu \in \mathbb{K}$, and $\xi \in \mathbb{K}\setminus\{0\}$ extends to an automorphism $\gamma : \mathcal{G}^{\mathrm{A}}_{\mathrm{m}}(\mathbb{K}) \to \mathcal{G}^{\mathrm{A}}_{\mathrm{m}}(\mathbb{K})$ of order 3 corresponding to the graph automorphism $\bar\gamma$



of the Dynkin diagram of A. Now, suppose that $\mathbb{K}$ is a field containing a primitive cube root of unity, say ϵ, and such that $\operatorname{char}\mathbb{K} \neq 2, 3$. We also show that the map

$$\begin{array}{lcll} x_{\alpha_0}(\mu) & \mapsto & x_{\alpha_0}(\epsilon\mu) & \\ x_{-\alpha_0}(\mu) & \mapsto & x_{-\alpha_0}(\epsilon^2\mu) & \\ x_{\pm\alpha_i}(\mu) & \mapsto & x_{\pm\alpha_i}(\mu) & \text{for } i \in \underline{4} \\ h_{\varpi^\vee_{\alpha_i}}(\xi) & \mapsto & h_{\varpi^\vee_{\alpha_i}}(\xi) & \text{for } i \in \underline{4}_0 \end{array}$$

for $\mu \in \mathbb{K}$, $\xi \in \mathbb{K}\setminus\{0\}$ extends to a diagonal automorphism $\tau : \mathcal{G}^{\mathrm{A}}_{\mathrm{m}}(\mathbb{K}) \to \mathcal{G}^{\mathrm{A}}_{\mathrm{m}}(\mathbb{K})$.

For each $i \in \underline{4}_0$, define $n_{\alpha_i} = x_{\alpha_i}(1)x_{-\alpha_i}(1)x_{\alpha_i}(1)$ and, for each $\mu \in \mathbb{K}$, let

$$\begin{array}{lclclcl} x_{\alpha_0+\alpha_1+\alpha_2}(\mu) & = & n_{\alpha_0}n_{\alpha_2}x_{\alpha_1}(\mu)n_{\alpha_2}^{-1}n_{\alpha_0}^{-1}, & x_{\alpha_0+\alpha_2+\alpha_3}(\mu) & = & n_{\alpha_0}n_{\alpha_2}x_{\alpha_3}(\mu)n_{\alpha_2}^{-1}n_{\alpha_0}^{-1}, \\ x_{\alpha_0+\alpha_2+\alpha_4}(\mu) & = & n_{\alpha_0}n_{\alpha_2}x_{\alpha_4}(\mu)n_{\alpha_2}^{-1}n_{\alpha_0}^{-1}, & x_{-\alpha_0-\alpha_1-\alpha_2}(\mu) & = & n_{\alpha_0}n_{\alpha_2}x_{-\alpha_1}(\mu)n_{\alpha_2}^{-1}n_{\alpha_0}^{-1}, \\ x_{-\alpha_0-\alpha_2-\alpha_3}(\mu) & = & n_{\alpha_0}n_{\alpha_2}x_{-\alpha_3}(\mu)n_{\alpha_2}^{-1}n_{\alpha_0}^{-1}, & x_{-\alpha_0-\alpha_2-\alpha_4}(\mu) & = & n_{\alpha_0}n_{\alpha_2}x_{-\alpha_4}(\mu)n_{\alpha_2}^{-1}n_{\alpha_0}^{-1}. \end{array}$$

The map

$$\begin{array}{lcl} y_{\beta_0}(\mu) & \mapsto & x_{\alpha_0+\alpha_2+\alpha_4}(\mu)x_{\alpha_0+\alpha_1+\alpha_2}(\epsilon\mu)x_{\alpha_0+\alpha_2+\alpha_3}(\epsilon^2\mu) \\ y_{-\beta_0}(\mu) & \mapsto & x_{-\alpha_0-\alpha_2-\alpha_4}(\mu)x_{-\alpha_0-\alpha_1-\alpha_2}(\epsilon\mu)x_{-\alpha_0-\alpha_2-\alpha_3}(\epsilon^2\mu) \\ y_{\pm\beta_1}(\mu) & \mapsto & x_{\pm\alpha_1}(\mu)x_{\pm\alpha_3}(\mu)x_{\pm\alpha_4}(\mu) \\ y_{\pm\beta_2}(\mu) & \mapsto & x_{\pm\alpha_2}(\mu) \\ h_{\varpi^\vee_{\beta_0}}(\xi) & \mapsto & h_{\varpi^\vee_{\alpha_0}}(\xi) \\ h_{\varpi^\vee_{\beta_1}}(\xi) & \mapsto & h_{\varpi^\vee_{\alpha_0}}(\xi^{-1})h_{\varpi^\vee_{\alpha_1}}(\xi)h_{\varpi^\vee_{\alpha_3}}(\xi)h_{\varpi^\vee_{\alpha_4}}(\xi) \\ h_{\varpi^\vee_{\beta_2}}(\xi) & \mapsto & h_{\varpi^\vee_{\alpha_0}}(\xi^{-1})h_{\varpi^\vee_{\alpha_2}}(\xi) \end{array}$$

for $\mu \in \mathbb{K}$ and $\xi \in \mathbb{K}\setminus\{0\}$, extends to an isomorphism $\Psi_m(\mathbb{K}) : \mathcal{G}^{\mathrm{B}}_{\mathrm{m}}(\mathbb{K}) \to \mathcal{G}^{\gamma;\tau}_{\mathrm{m}}(\mathbb{K})$. We note that under the action of $\Psi_m(\mathbb{K})$

$$\begin{array}{lcl} h_{\beta_0}(\xi) & \mapsto & h_{\alpha_0}(\xi^3)h_{\alpha_2}(\xi^3)h_{\alpha_1}(\xi)h_{\alpha_3}(\xi)h_{\alpha_4}(\xi) \\ h_{\beta_1}(\xi) & \mapsto & h_{\alpha_1}(\xi)h_{\alpha_3}(\xi)h_{\alpha_4}(\xi) \qquad\qquad \text{and} \\ h_{\beta_2}(\xi) & \mapsto & h_{\alpha_2}(\xi) \end{array}$$

for all $\xi \in \mathbb{K}\setminus\{0\}$.

References

[1] R. Carter and Y. Chen. Automorphisms of affine Kac-Moody groups and related Chevalley groups over rings. *J. of Algebra*, 155:44–94, 1993.
[2] J.-Y. Hée. Construction de groupes tordus en théorie de Kac-Moody. *C.R. Acad. Sci. Paris*, 310:77–80, 1990.
[3] J.-Y. Hée. Torsion de groupes munis d'une donnée radicielle. March 1991.
[4] V. Kac. *Infinite Dimensional Lie Algebras.* CUP, Cambridge, third edition, 1990.

[5] J. Ramagge. On certain fixed point subgroups of affine Kac-Moody groups. To appear in the *J. of Algebra*.

[6] J. Ramagge. A realization of certain affine Kac-Moody groups of types II and III. To appear in the *J. of Algebra*.

[7] J. Ramagge. On some twisted Kac-Moody groups. Preprint 50/1992, Warwick University, September 1992.

[8] R. Steinberg. Lectures on Chevalley groups. Yale University Mathematics Department, 1967.

[9] J. Tits. Uniqueness and presentation of Kac-Moody groups over fields. *J. of Algebra*, 105:542–573, 1987.

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