An Introduction to Kac-Moody Groups

Jacqui Ramagge*

Background

In the mid 1950's, Chevalley and Demazure defined 'analogues' over arbitrary fields of the complex simple Lie groups. The notion was successively refined and culminated with the introduction in the mid 1960's of affine groups schemes.

Thus, to every reductive group G over $\mathbf{C}$ is associated a group functor $\mathcal{G}$ on the category of all commutative rings such that $\mathcal{G}(\mathbf{C}) = G$. Such functors are known as *Chevalley-Demazure Group Schemes* and are characterized by a few simple properties.

Reductive groups over $\mathbf{C}$, and hence Chevalley-Demazure group schemes, are classified by *root data*

$$\mathcal{D} = \left(\underline{n} \, , \Lambda \, , \{\alpha_i\}_{i\in\underline{n}} \, , \{\alpha_i^\vee\}_{i\in\underline{n}} \right)$$

consisting of:

- a finite set $\underline{n}$ (essentially indexing the conjugacy classes of the maximal parabolics),
- a finitely generated free abelian group Λ (corresponding to the character group of a maximal torus),
- a subset $\{\alpha_i\}_{i\in\underline{n}}$ of Λ (a basis of the root system), and
- a subset $\{\alpha_i^\vee\}_{i\in\underline{n}}$ of Λ^*, the $\mathbf{Z}$-dual of Λ (a basis of the coroot system),

subject to the condition that

$$A = (A_{ij})_{i,j\in\underline{n}} := \left(\alpha_i^\vee(\alpha_j)\right)_{i,j\in\underline{n}}$$

be a Cartan matrix.

Recall that A is a Cartan matrix if

$$(GCM) \qquad \begin{cases} A_{ii} = 2, A_{ij} \leq 0 \text{ if } i \neq j, \\ A_{ij} = 0 \Leftrightarrow A_{ji} = 0, \\ A = DS \text{ where } D \text{ is an invertible diagonal complex matrix, and} \\ \ S \text{ is a symmetric complex matrix, and} \end{cases}$$

$$(Pos) \qquad S \text{ is positive definite.}$$

The Cartan matrix A also determines the Lie algebra associated to G via Serre's presentation. Note that the first three conditions mean that our fundamental reflections (when we come to define them) are indeed reflections, the symmetrizability condition allows us to define a Killing form, and condition (Pos) ensures that the Lie algebra obtained is finite-dimensional.

The theory of Kac-Moody algebras and Kac-Moody groups can be viewed as the study of Lie algebras and groups associated to root data $\mathcal{D}$ obtained when A satisfies the conditions (GCM) but not necessarily the condition (Pos). In fact some of the theory has been developed

* Winner of the B. H. Neumann Prize for the most outstanding talk by a student at the 1992 annual meeting of the Australian Mathematical Society. This article is based on her prize winning talk.

even when the symmetrizability condition does not hold, but then we can not define an analogue of the Killing form.

The Kac-Moody algebra $\mathbf{g}(A)$

Given a matrix $A = (A_{ij})_{i,j\in\underline{n}}$ of rank r, with integer entries and satisfying (GCM), we construct a triple

$$(\mathbf{h}, \Pi, \Pi^\vee)$$

such that

- $\mathbf{h} \cong \mathbf{C}^{2n-r}$,
- $\Pi^\vee = \{\alpha_i^\vee\}_{i\in\underline{n}}$ is a set of n independent elements in $\mathbf{h}$,
- $\Pi = \{\alpha_i\}_{i\in\underline{n}}$ is a set of n independent elements in $\mathbf{h}^*$, and
- $\alpha_j(\alpha_i^\vee) = A_{ij}$ for all $i, j \in \underline{n}$.

Such a triple always exists and is unique up to isomorphism. Furthermore, two matrices give rise to isomorphic triples if and only if one can be obtained from the other by a permutation of the indexing set.

Let $Q(A) = \sum_{i\in\underline{n}} \mathbf{Z}\alpha_i$. We define the *(complex) Kac-Moody algebra*, $\mathbf{g}(A)$, by the (Lie) presentation

- generators: e_{α_i}, f_{α_i}, $i \in \underline{n}$, $h \in \mathbf{h}$,
- relations: Serre's relations.

As in the classical case, we define

$$\mathbf{g}_\alpha = \{x \in \mathbf{g}(A) : [h, x] = \alpha(h)x, \text{ for all } h \in \mathbf{h}\}$$

for each $\alpha \in \mathbf{h}^*$. Then, as a result of a theorem proved independently by Kac and Moody, $\mathbf{g}(A)$ has a *vector space decomposition with respect to* $\mathbf{h}$ given by

$$\mathbf{g}(A) = \bigoplus_{\alpha\in Q} \mathbf{g}_\alpha$$

and such that

$$\mathbf{g}_0 = \mathbf{h} \text{ and } \dim \mathbf{g}_\alpha < \infty \text{ for all } \alpha \in Q.$$

We call α a *root* if $\alpha \neq 0$ and $\dim \mathbf{g}_\alpha \neq 0$. Note that if $\mathbf{g}(A)$ is infinite dimensional this means there are an infinite number of roots. Denote by $\Phi(A)$ the set of roots of $\mathbf{g}(A)$.

To each element $\alpha_i \in \Pi$ we associate an element $r_{\alpha_i} \in GL(\mathbf{h}^*)$ given by

$$r_{\alpha_i}(\beta) = \beta - \beta(\alpha_i^\vee)\alpha_i$$

for each $\beta \in \mathbf{h}^*$ and called the *fundamental reflection corresponding to* α_i. We define the *Weyl group*, $W(A)$, of $\mathbf{g}(A)$ to be the subgroup of $GL(\mathbf{h}^*)$ generated by the elements $\{r_{\alpha_i}\}_{i\in\underline{n}}$. The Weyl group turns out to be a Coxeter group with the $\{r_{\alpha_i}\}_{i\in\underline{n}}$ as Coxeter generators. Note further that $W(A)$ acts on $\mathbf{h}$ by duality.

We call a root $\alpha \in \Phi(A)$ *real* if there exist elements $\alpha_i \in \Pi$ and $w \in W(A)$ such that

$$\alpha = w(\alpha_i).$$

Note that in the classical theory, that is when A satisfies (Pos), all roots are real. Let $\Phi^{re}(A)$ denote the set of real roots. These have all the properties we associate with roots in the classical theory, eg. $\dim \mathbf{g}_\alpha = 1$, $k\alpha \in \Phi^{re}(A) \Leftrightarrow k = \pm 1$, and so on.

In the general case, however, there are roots which can not be expressed in this manner. We call such roots *imaginary roots* and we denote the set of imaginary roots by $\Phi^{im}(A)$. Their behaviour is very different to that of real roots and in particular

$$\delta \in \Phi^{im}(A) \Rightarrow m\delta \in \Phi^{im}(A) \text{ for all } m \in \mathbf{Z}.$$

Kac-Moody Groups Associated to A

Recall that there are essentially two equivalent ways of describing Chevalley groups, namely

- via Chevalley-Demazure group schemes (so as group functors on the category of commutative rings), and
- via Steinberg's presentation.

The natural question then arises: is something similar true of Kac-Moody groups? Tits has shown that to a certain extent something similar does happen.

In 1987 a fundamental paper by Tits appeared in which he

- gave a set of axioms for a Kac-Moody group functor which naturally generalized the notion of a Chevalley-Demazure group scheme,
- constructed a group functor on the category of commutative rings $\mathcal{G}_{\mathcal{D}}$ based on a presentation analogous to that given by Steinberg for Chevalley groups, but using only $\Phi^{re}(A)$, and
- proved that these two functors coincided over the category of fields.

I am not aware of any further progress over arbitrary commutative rings.

All Kac-Moody groups are endowed with at least one $(B\,,N)$-pair, and thus have all the properties which are associated with such a structure. In fact Kac-Moody groups corresponding to affine matrices (ie. matrices with zero determinant but all of whose proper principal minors are positive) have twin $(B\,,N)$-pairs.

Twisted Kac-Moody Groups

The twisted Chevalley groups are essentially the fixed point subgroups of certain Chevalley groups under particular automorphisms. In the late 1950's, Hertzig, Steinberg and Tits initiated a study of the twisted Chevalley groups which culminated in the construction of the Suzuki and Ree groups in the early 1960's.

An analogous theory for Kac-Moody groups has been developed by Hée. Though far-reaching, Hée's approach is not quite comprehensive. This is because some Kac-Moody groups admit automorphisms whose behaviour has no analogy in the case of Chevalley groups. This is essentially due to the existence of imaginary roots.

Suppose we begin with a Kac-Moody group G associated to a GCM A and consider an automorphism σ of G. In the case of Chevalley groups and affine Kac-Moody groups we know that σ decomposes into a product of basic automorphisms and we denote by γ the graph automorphism constituent of σ. In general, the roots of the twisted root system correspond to orbits of roots in $\Phi^{re}(A)$. Let us denote the twisted real root system by Φ^{γ} and suppose $\beta \in \Phi^{\gamma}$ corresponds to an orbit $\Phi_{\beta} \subseteq \Phi^{re}(A)$. We define a subgroup $X_{\beta} \subseteq G^{\sigma}$ associated to $\beta \in \Phi^{\gamma}$ by letting

$$X_{\beta} = \langle X_{\alpha} : \alpha \in \Phi_{\beta} \rangle^{\sigma}.$$

In the case of Chevalley groups, the subgroups X_{β} are non-trivial for all $\beta \in \Phi^{\gamma}$ and in fact play the role of root subgroups in G^{σ}. However, in the case of affine Kac-Moody groups there are automorphisms for which $X_\beta = \{1\}$ for some $\beta \in \Phi^\gamma$. Thus, in these cases, the root system of G^σ is different from Φ^γ. My doctoral thesis considers the fixed point subgroups of some affine Kac-Moody groups under such automorphisms.

References

[CC] R. W. Carter and Y. Chen. Automorphisms of affine Kac-Moody groups and related Chevalley groups over rings. *Warwick University Preprints*: April 1991.

[C1] C. Chevalley. Certain schémas de groupes semi-simples. *Seminaire Bourbaki*, 13, 1960. Exposé 219.

[C2] C. Chevalley. La théorie des groupes algébriques. *Proceedings of the International Congress of Mathematicians, Edinburgh*, Cambridge University Press (1960).

[DG] M. Demazure and A. Grothendieck. Schémas en groupes, I, II, III. *Lecture Notes in Mathematics*, 151, 152, 153, Springer.

[H] J-Y. Hée. Construction de groupes tordus en thèorie de Kac-Moody. *Comptes Rendu de Academie Scientifique de Paris*, 310: 77–80, 1990.

[He] D. Hertzig. On simple algebraic groups. *Proceedings of the International Congress of Mathematicians, Edinburgh*, Cambridge University Press (1960).

[K] V. G. Kac. *Infinite Dimensional Lie Algebras.* Cambridge University Press, Cambridge, third edition, 1990.

[M] R. V. Moody. Lie algebras associated with generalized Cartan matrices. *Bulletin of the American Mathematical Society*, 73:271–221, 1967.

[R] J. Ramagge. On some twisted Kac-Moody groups. *Warwick University Preprints*: 50/1992, September 1992.

[R1] R. Ree. A Family of simple groups associated with the simple Lie algebra of type (F_4). *American Journal of Mathematics*, 83: 401–420, 1961.

[R2] R. Ree. A Family of simple groups associated with the simple Lie algebra of type (G_2). *American Journal of Mathematics*, 83: 432–462, 1961.

[S] R. Steinberg. Lectures on Chevalley groups. Mimeographed lecture notes, Yale University Mathematics Department, 1967.

[Su] M. Suzuki. A new type of simple groups of finite order. *Proc. Nat. Acad. Sci. U.S.A.*, 46 (1960) 868–870.

[T] J. Tits. Uniqueness and presentation of Kac-Moody groups over fields. *Journal of Algebra*, 105:542–573, 1987.

[W] W. Waterhouse. *Introduction to Affine Group Schemes*, volume 66 of *Graduate Texts in Mathematics*. Springer-Verlag, 1979.

Department of Mathematics
University of Newcastle
NSW 2308