Hecke algebras from groups acting on trees and HNN extensions ☆

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Abstract

We study Hecke algebras of groups acting on trees with respect to geometrically defined subgroups. In particular, we consider Hecke algebras of groups of automorphisms of locally finite trees with respect to vertex and edge stabilizers and the stabilizer of an end relative to a vertex stabilizer, assuming that the actions are sufficiently transitive. We focus on identifying the structure of the resulting Hecke algebras, give explicit multiplication tables of the canonical generators and determine whether the Hecke algebra has a universal C^*-completion. The paper unifies algebraic and analytic approaches by focusing on the common geometric thread. The results have implications for the general theory of totally disconnected locally compact groups.
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Introduction

In this paper we study Hecke algebras associated with groups acting on locally finite trees. Hecke algebras are a representation-theoretic tool. Just as representations of a group G on a complex vector space correspond to representations of its group algebra $\mathbb{C}[G]$, the algebra representations of the Hecke algebra $\mathbb{C}[G, G_0]$ of G relative to an almost normal subgroup G_0 correspond to linear representations of G which are generated by their G_0-fixed vectors. In the context of an arbitrary subgroup G_0, such representations are called *smooth* or *algebraic* representations of the group G, see [5,10,18] and [20, Chapter 1]. The groups G considered here are subgroups of the automorphism group of a tree, with the almost normal subgroups G_0 being either the stabilizer of a vertex or the pointwise stabilizer of an edge of the tree.

Groups of tree automorphisms that do not fix an end of the tree are treated first. It is seen that in fact all groups G of tree automorphisms that are sufficiently highly transitive yield the same Hecke algebra in each of the cases when G_0 is the stabilizer of a vertex and the pointwise stabilizer of an edge. In the vertex stabilizer case, this Hecke algebra is the algebra of radial functions on the tree, as already calculated in [25] and [15].

Groups of automorphisms that do fix an end are of particular importance. These occur as HNN-extensions through Bass–Serre theory [29] and, essentially for that reason, they play a role in the structure theory of totally disconnected locally compact groups analogous to that filled by the $(ax + b)$-group in Lie theory, see [32, Proposition 2], [33] and the tree representation theorem [4, Theorem 4.1]. The representation theory of these groups is therefore basic to the representation theory of general totally disconnected groups. In this case the Hecke algebra depends very much on which subgroup G of the full group of automorphisms fixing the end is considered. We initiate the study of these algebras by computing the structure of a few examples in the last section, giving explicit multiplication tables of the canonical generators.

In the functional analytic setting, it is unitary representations on Hilbert space that are of importance and these correspond to representations of a C^*-algebra. Inversion of group elements gives rise to a *-algebraic structure on a Hecke algebra, and we are also able to decide in some of our examples whether the Hecke *-algebra has an enveloping C^*-algebra, a question that has recently attracted a lot of interest, see [2,3,6,7,17,21,22,31]. We use general results on Hecke algebras for semidirect products to show that our Hecke algebras and C^*-algebras are semigroup crossed products. We also indicate how to use the Pimsner–Voiculescu six-term exact sequence to compute their K-theory.

We point out that some of the Hecke algebras presented here are essentially known at the algebraic level. Indeed, the multiplication formulas in our Theorem 3.3 have been described elsewhere, see Remark 3.6 below. With a modicum of translation the formulas in our Theorem 4.4(3) appear in [19, Section 3] and also, for more general buildings, in [27, Theorem 3.4]. A reduction like the one presented in our Theorem 4.4(3)(b) expressing a Hecke algebra as a twisted tensor product already appears in [19, Proposition 3.8]. With respect to this, it is interesting to note that, in the context of HNN extensions, our Theorem 5.2 below can be regarded as a generalization of this decomposition, in which the tensor product with the group algebra twisted by the induced automorphism group is replaced by the (semigroup) crossed product by the induced action of the semigroup $\mathbb{N}$ by endomorphisms.

1. Notation and basic facts concerning Hecke algebras

Suppose that G is a group and H is a subgroup of G. Then (G, H) is a *Hecke pair* if and only if H is *almost normal* in G, that is, it satisfies the following three equivalent conditions:

- For any $g \in G$, the index of $H \cap gHg^{-1}$ in H and gHg^{-1} is finite, in other words, G equals the commensurator subgroup of H in G, $\{x \in G: x \text{ commensurates } H\}$;
- Every H double coset contains finitely many right H cosets;
- Every H double coset contains finitely many left H cosets.

That (G, H) is a Hecke pair is sufficient to ensure that the product of any two H double cosets is a finite union of H double cosets. This condition also allows to define an associative algebra, whose rule of multiplication for two H double cosets is a 'weighted version' of the formula for the set-theoretic product of those double cosets.

Definition 1.1 (*Hecke algebra over a ring*). Suppose that (G, H) is a Hecke pair. The *Hecke algebra* $\mathbb{Z}[G, H]$ of (G, H) over the ring of integers $\mathbb{Z}$ is the free $\mathbb{Z}$-module with basis the set $H \backslash G / H$ of double cosets of H in G endowed with the multiplication defined by

$$HgH \cdot Hg'H = \sum_{x \in G} \alpha(g, g'; x) HxH,$$

where $HgH = \bigsqcup_i Hg_i$, $Hg'H = \bigsqcup_j Hg'_j$ and $\alpha(g, g'; x) := \#\{(i, j): g_i g'_j \in Hx\}$. Also, for any ring $\mathcal{R}$, the *Hecke algebra of (G, H) over $\mathcal{R}$* is $\mathcal{R}[G, H] := \mathbb{Z}[G, H] \otimes_{\mathbb{Z}} \mathcal{R}$. See [20, Section 4] for the details. When $\mathbb{Z}[G, H]$ is commutative, (G, H) is called a *Gelfand pair*.

When the ring $\mathcal{R}$ is the field of complex numbers, Hecke algebras can be endowed with a conjugate linear involution $*: \mathbb{C}[G, H] \to \mathbb{C}[G, H]$ defined on scalar multiples of the canonical basis by $(\lambda HxH)^* := \bar{\lambda} Hx^{-1}H$ and turning $\mathbb{C}[G, H]$ into a unital *-algebra. If the (unital, *-preserving) representations of $\mathbb{C}[G, H]$ on Hilbert space give rise to a C^*-norm on $\mathbb{C}[G, H]$, we define the universal enveloping C^*-algebra $C^*(G, H)$ to be the completion of $\mathbb{C}[G, H]$ with respect to this norm. The existence of $C^*(G, H)$ is not automatic and we shall be interested in classifying our examples according to it.

Note that every normal subgroup of G is almost normal and the Hecke algebra with respect to a normal subgroup is the group algebra of the quotient group. In this sense Hecke algebras are an alternative to group algebras of quotients and they play an analogous representation-theoretic role.

We denote the finite numbers of right respectively left cosets contained in a double coset by

$$R(HxH) := \#H \backslash HxH, \qquad L(HxH) := \#HxH/H = R(Hx^{-1}H).$$

We usually abuse notation and write $R(x) := R(HxH)$ and $L(x) := L(HxH)$. On many occasions we make use of Corollary 4.5 in [20], which states that, for any ring $\mathcal{R}$, the function R extends to an $\mathcal{R}$-algebra homomorphism $R: \mathcal{R}[G, H] \to \mathcal{R}$.

2. Notation and basic facts concerning trees and their automorphism groups

Throughout this paper $\mathcal{T}$ will denote an infinite tree with vertex set $\mathcal{T}^0$ and edge set $\mathcal{T}^1$. A tree is a bipartite graph, that is, we have a decomposition $\mathcal{T}^0 = \mathcal{T}_{\text{even}} \cup \mathcal{T}_{\text{odd}}$ with the property that each vertex in one subset is connected only to vertices in the other subset. Vertices in the same subset are said to have the same *type* and automorphisms of the tree which preserve this decomposition are said to be *type-preserving*.

There is a graph-theoretic distance function $d : \mathcal{T}^0 \times \mathcal{T}^0 \to \mathbb{N}$ on the vertices of the tree. In cases where we use a distinguished vertex in our arguments, we choose our notation so that $\mathcal{T}_{\mathrm{even}}$ is the set of vertices at even distance from our distinguished vertex and, consequently, $\mathcal{T}_{\mathrm{odd}}$ is the set of vertices at odd distance from our distinguished vertex. Given a group G of automorphisms of $\mathcal{T}$, the stabilizer of a distinguished vertex will be denoted G_0.

By a *geodesic line* in a tree we will mean a subgraph isomorphic to the real line, $\mathbb{R}$, triangulated by the vertex set $\mathbb{Z}$ of integers. A *ray* in a tree is a subgraph isomorphic to the half-line of positive reals, triangulated by the set of non-negative integers. An *end* of a tree is an equivalence class of rays in the tree, where two rays are equivalent if their intersection is a ray.

We make constant use of the classification of automorphisms of a tree obtained by Tits in Proposition 3.2 of [30], for which Section 6.4 in [29] is also a good reference. An automorphism of a tree either (1) fixes a vertex; (2) inverts an edge (that is, exchanges its two vertices); or (3) has a (unique) invariant geodesic line, called its *axis*, on which it induces a translation of non-zero amplitude, which is called the *translation length* of the automorphism. The possibilities (1)–(3) above are mutually exclusive and an automorphism satisfying (1) or (2) is called *elliptic* while an automorphism satisfying (3) is called *hyperbolic*. Elliptic automorphisms of type (2) are called *inversions* and the requirement that a group acts on a tree *without inversion* means that its elements act only as automorphisms of type (1) or (3). Of the two ends defined by rays contained in the axis of a hyperbolic automorphism, h say, one has the property that every ray contained in the axis in that end is mapped into itself by h; this end is called *attracting for h* while the other end is called *repelling for h*.

We will be working with trees which turn out to be *semi-homogeneous* as a consequence of the conditions imposed on their automorphism groups. This means that vertices of a given type are incident on the same number of edges and we call this number the *vertex degree* or *ramification index* of that type of vertex. The formulae are more elegant if the ramification indices of the even and odd types are denoted $q_0 + 1$ and $q_1 + 1$ respectively. If $q_0 = q_1$, the tree is *homogeneous*. A group can only act transitively on the vertex set $\mathcal{T}^0$ if $\mathcal{T}$ is homogeneous.

We shall use the basic facts collected in the following lemma several times during our analysis of Hecke algebras; they relate algebraic properties of the Hecke algebra to geometric properties of the group action.

Lemma 2.1. *Suppose the group Γ acts by isometries on the pseudo-metric space X and let o be a point of X with stabilizer Γ_0 in Γ. Then the following statements hold for the pair (Γ, Γ_0).*

(1) *The number $L(g)$ is the cardinality of the orbit of $g \cdot o$ under Γ_0.*
(2) *If the sphere with center o and radius $d(o, g \cdot o)$ is finite, then $L(g)$ is finite. In particular, if all balls in X are finite, then (Γ, Γ_0) is a Hecke pair.*
(3) *If the action of Γ_0 on the sphere with center o and radius $d(o, g \cdot o)$ is transitive, then $\Gamma_0 g \Gamma_0$ is the set of elements of Γ mapping o to some element on the sphere with center o and radius $d(o, g \cdot o)$. In particular $L(g) = R(g)$ is the cardinality of that sphere and $\Gamma_0 g^{-1} \Gamma_0 = \Gamma_0 g \Gamma_0$ is self adjoint in the Hecke algebra $\mathbb{C}[\Gamma, \Gamma_0]$.*
(4) *If all balls in X are finite and the action of Γ_0 on all spheres with center o is transitive, then (Γ, Γ_0) is a Gelfand pair, i.e. $\mathbb{C}[\Gamma, \Gamma_0]$ is commutative.*

Proof. First note that for all $g_1, g_2 \in \Gamma$

$$g_1 \cdot o = g_2 \cdot o \quad \Leftrightarrow \quad g_2^{-1} g_1 \cdot o = o \quad \Leftrightarrow \quad g_2^{-1} g_1 \in \Gamma_0 \quad \Leftrightarrow \quad g_1 \in g_2 \Gamma_0.$$

Thus each left Γ_0-coset $g\Gamma_0$ corresponds to an element $g \cdot o$ in the Γ-orbit of o. Now (2.1) follows from the observation that $\Gamma_0 g \Gamma_0 = \Gamma_0(g \Gamma_0)$.

Since the action is by isometries on X, the orbit of $g \cdot o$ under Γ_0 is contained in the sphere with center o and radius $d(o, g \cdot o)$, hence (2.1) follows from (2.1). We now turn to (2.1). Since the action of Γ_0 on the sphere with center o and radius $d(o, g \cdot o)$ is transitive, $\Gamma_0 g \Gamma_0$ is the set of elements of Γ mapping o to some element on this sphere. In particular $d(g^{-1} \cdot o, o) = d(o, g \cdot o)$, so g^{-1} is such an element and $\Gamma_0 g^{-1} \Gamma_0 = \Gamma_0 g \Gamma_0$. This also implies that $L(g)$ and $R(g)$ are both equal to the cardinality of the sphere. Statement (2.1) now follows because the generators of $\mathbb{C}[\Gamma, \Gamma_0]$ are self-adjoint by (2.1), and thus satisfy $ab = a^*b^* = (ba)^* = ba$. $\quad\square$

Remark 2.2. In Lemma 2.1 we may assume that Γ acts transitively by taking $X = \Gamma \cdot o$.

Remark 2.3. Even in cases when the action of Γ_0 on spheres centered at o is not transitive, the triangle inequality imposes a restriction on the scattering resulting from the multiplication of double cosets. Indeed, suppose that Γ acts by isometries on a metric space X in which spheres are finite, and let A_n be the $\mathbb{C}$-linear span of all the double Γ_0-cosets of the elements $\gamma \in \Gamma$ such that $d(\gamma \cdot o, o) = n$ (with the convention that $A_n = \{0\}$ if no such elements exists). Then $A_n A_m \subseteq \bigoplus_{i=|m-n|}^{m+n} A_i$ in $\mathbb{C}[\Gamma, \Gamma_0]$.

## 3. Edge-transitive automorphism groups relative to a vertex stabilizer

In this section we will assume that the locally finite tree $\mathcal{T}$ has at least three edges at each vertex, and we select a distinguished vertex o of $\mathcal{T}$. We will also suppose that Γ is a group that acts on $\mathcal{T}$ in such a way that the stabilizer Γ_0 of o is a proper subgroup of Γ that acts transitively on each sphere about o. We will show that under these conditions the underlying tree $\mathcal{T}$ is necessarily semi-homogeneous and that such a pair (Γ, Γ_0) is a Gelfand pair for which the Hecke algebra is simply the algebra of polynomials in a self adjoint variable. The situation under consideration includes as special cases those in which Γ is the full group of automorphisms of a homogeneous tree, the subgroup of type-preserving automorphisms of a semi-homogeneous tree, and many of their proper subgroups, some of which cases have been studied before independently e.g. in [11]. More examples of the present situation arise naturally from semisimple matrix groups of rank 1 over local fields acting on their Bruhat–Tits trees, the simplest example being $SL_2(\mathbb{Q}_p)$, see Remark 3.6.

Proposition 3.1. *Let $\mathcal{T}$ be a tree with at least 3 edges at each vertex and let o be a distinguished vertex of $\mathcal{T}$. Suppose that Γ is a group acting by automorphisms of $\mathcal{T}$ and assume that the stabilizer Γ_0 of o is a proper subgroup of Γ that acts transitively on each sphere around o. Then the tree $\mathcal{T}$ is semi-homogeneous and the action of Γ on $\mathcal{T}^0$ is either transitive or else has two orbits, namely, the sets $\mathcal{T}_{\text{even}} = \{v \in \mathcal{T}^0 : d(v, o) \text{ is even}\}$ and $\mathcal{T}_{\text{odd}} = \{v \in \mathcal{T}^0 : d(v, o) \text{ is odd}\}$.*

Proof. By part (4) of Lemma 2.1 applied to the action of Γ on the subset $\mathcal{T}^0$, (Γ, Γ_0) is a Gelfand pair. In order to verify that the tree $\mathcal{T}$ is semi-homogeneous and that the action of Γ on the set $\mathcal{T}^0$ of vertices of $\mathcal{T}$ has at most 2 orbits, it is enough to show the following lemma. This will finish the proof of the proposition. $\quad\square$

Lemma 3.2. *For each even integer D there exists an elliptic element $\delta \in \Gamma$ with $d(\delta \cdot o, o) = D$.*

Proof. By [30, Proposition 3.4] Γ contains a hyperbolic element η. In particular, $d(\eta^n \cdot o, o)$ goes to infinity as n goes to infinity, and one can choose an element γ in Γ such that $d(\gamma \cdot o, o) \geqslant D$. Let v be the vertex at distance $D/2$ from $\gamma \cdot o$ on the path joining o to $\gamma \cdot o$. Let w be a vertex at distance $D/2$ from v on the sphere with radius $d(\gamma \cdot o, o)$ about o such that $[\gamma \cdot o, o] \cap [o, w] \cap [w, \gamma \cdot o] = \{v\}$. Such a vertex w exists because there are at least 3 edges at each vertex by assumption. The situation is illustrated in the following diagram; note that because of our choice of w we have $d(w, \gamma \cdot o) = D$.
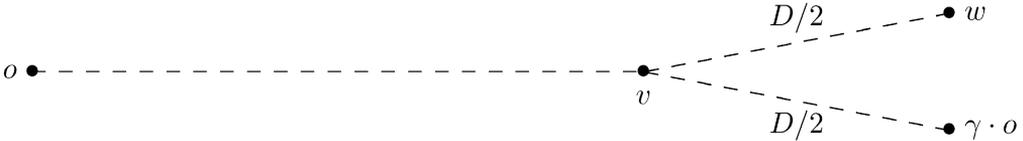
Choose an element π of Γ_0 mapping $\gamma \cdot o$ to w; such an element exists because Γ_0 acts transitively on the sphere with center o and radius $d(\gamma \cdot o, o)$. The element $\delta := \gamma^{-1}\pi\gamma$ is elliptic, because π is, and since

$$d(\delta \cdot o, o) = d(\gamma^{-1}\pi\gamma \cdot o, o) = d(\gamma^{-1} \cdot w, o) = d(w, \gamma \cdot o) = D,$$

this finishes the proof of the lemma and of Proposition 3.1. □

Theorem 3.3. *Under the same assumptions as in Proposition* 3.1, *let* $q_0 + 1$ *(respectively* $q_1 + 1$*) denote the degrees of the even (respectively odd) vertices and put* $q := q_0$ *if* $\mathcal{T}$ *is homogeneous. Define* $\Gamma_n := \{\gamma \in \Gamma : d(\gamma \cdot o, o) = n\}$. *Then*

(1) *in the case when* Γ *acts transitively on vertices, the canonical basis of* $\mathbb{C}[\Gamma, \Gamma_0]$ *is the set* $\{\Gamma_n : n \in \mathbb{N}\}$, *the map* $T \mapsto \Gamma_1$ *extends to an isomorphism of the algebra* $\mathbb{C}[T]$ *of complex polynomials in the variable* T *onto* $\mathbb{C}[\Gamma, \Gamma_0]$, *and the elements of the canonical basis of* $\mathbb{C}[\Gamma, \Gamma_0]$ *satisfy the recursion relations*

$$\Gamma_1\Gamma_n = (q + \delta_{n,1})\Gamma_{n-1} + \Gamma_{n+1} \quad \textit{for } n \geqslant 1;$$

(2) *in the case when* Γ *has two orbits of vertices, the canonical basis is the set* $\{\Gamma_{2n} : n \in \mathbb{N}\}$, *the isomorphism of* $\mathbb{C}[T]$ *to* $\mathbb{C}[\Gamma, \Gamma_0]$ *is determined by* $T \mapsto \Gamma_2$ *and the recursion relations are*

$$\Gamma_2\Gamma_{2n} = (q_0 + \delta_{n,1})q_1\Gamma_{2n-2} + (q_1 - 1)\Gamma_{2n} + \Gamma_{2n+2} \quad \textit{for } n \geqslant 1.$$

Proof. That the given sets are canonical bases is immediate from parts (3) and (4) of Lemma 2.1 applied to the action of Γ on the set $\mathcal{T}^0$ of vertices. To show that $T \mapsto \Gamma_1$ and $T \mapsto \Gamma_2$ give isomorphisms we need the following lemma.

Lemma 3.4. *If both* Γ_n *and* Γ_m *are non-empty, then there exist* $\gamma_n \in \Gamma_n$ *and* $\gamma_m \in \Gamma_m$ *such that* $\gamma_n\gamma_m \in \Gamma_{n+m}$.

Proof. Considering the inverse of the product $\gamma_n \gamma_m$ if necessary, we may suppose that $n \leqslant m$. Choose γ_m in Γ_m arbitrarily. Suppose first that Γ_n has an elliptic element, say δ_n. Then by transitivity of the action of Γ_0 on the sphere of radius n around o, a suitable conjugate γ_n by an element of Γ_0 satisfies $\gamma_n \cdot o \notin [o, \gamma_m]$. With this choice of γ_n we have $d(\gamma_n \gamma_m \cdot o, o) = n + m$. Alternatively, suppose Γ_n has a hyperbolic element, say δ_n. Then, again by transitivity of the action of Γ_0 on the sphere of radius n around o, there is a Γ_0-conjugate, γ_n say, of δ_n such that the closest point on its axis to o lies on $[o, \gamma_m \cdot o]$ and o does not lie between $\gamma_n . o$ and $\gamma_m . o$. In this case, this choice of γ_n guarantees $d(\gamma_n \gamma_m \cdot o, o) = n + m$, finishing the proof of Lemma 3.4. $\quad\square$

We continue the proof of Theorem 3.3. Let n and m be integers with $n \leqslant m$. By Lemma 3.4 and Remark 2.3 we have

$$\Gamma_n \Gamma_m = \sum_{i=m-n}^{m+n} \lambda_i \Gamma_i; \quad \lambda_{m+n} \neq 0. \tag{3.1}$$

If Γ is transitive on the set of vertices of $\mathcal{T}$, we use this equation with $n = 1$ and arbitrary $m \geqslant 1$. If Γ has two orbits of vertices, then Γ_m is empty for m odd and we use this equation with $n = 2$ and m even and at least 2.

Since $\mathbb{C}[\Gamma, \Gamma_0]$ is abelian, (3.1) implies that each element of $\mathbb{C}[\Gamma, \Gamma_0]$ is a polynomial in Γ_i, with $i = 0$ if Γ acts transitively on the vertex set $\mathcal{T}^0$ and $i = 2$ if the action of Γ on $\mathcal{T}^0$ has two orbits. This implies that the map $T \mapsto \Gamma_i$ extends to an algebra homomorphism of the algebra $\mathbb{C}[T]$ of polynomials on T onto $\mathbb{C}[\Gamma, \Gamma_0]$. Since every proper quotient of $\mathbb{C}[T]$ is a finite-dimensional $\mathbb{C}$-vector space, and both canonical bases are infinite we have an isomorphism in both cases.

Next we verify the recursion relations. If Γ acts transitively on $\mathcal{T}^0$, then the set-theoretic product of the double cosets Γ_1 and Γ_n is $\Gamma_{n-1} \cup \Gamma_{n+1}$. Then

$$\Gamma_1 \Gamma_n = a\Gamma_{n-1} + b\Gamma_{n+1}$$

with both a and b positive integers. On applying the algebra homomorphism R to this equation, noting that $R(\Gamma_0) = 1$ and $R(\Gamma_n) = (q+1)q^{n-1}$ for $n \geqslant 1$ by part (3) of Lemma 2.1, we obtain the following equations for the cases $n = 1$ and $n > 1$:

$$(q+1)^2 q^{n-1} = \begin{cases} a + b(q+1)q & \text{if } n = 1; \\ a(q+1)q^{n-2} + b(q+1)q^n & \text{if } n > 1. \end{cases}$$

In these equations we cannot have $b \geqslant 2$ because then $b(q+1)q^n \geqslant 2(q+1)q^n$, which is strictly larger than the left-hand side $(q+1)^2 q^{n-1}$, whatever the value of n; thus we conclude that $b = 1$. This enables us to solve the equations for the parameter a in both cases, from which we obtain $a = q + 1$ when $n = 1$ and $a = q$ when $n > 1$.

If Γ acts with two orbits on $\mathcal{T}^0$, then the set-theoretic product of the double cosets Γ_2 and Γ_{2n} is the set $\Gamma_{2n-2} \cup \Gamma_{2n} \cup \Gamma_{2n+2}$. Then

$$\Gamma_2 \Gamma_{2n} = a\Gamma_{2(n-1)} + b\Gamma_{2n} + c\Gamma_{2(n+1)}$$

with a, b, and c positive integers. Applying again the homomorphism R using part (3) of Lemma 2.1 and the cardinality of the spheres of a semi-homogeneous tree in this case, we see that

$$(q_0 + 1)^2 q_1^2 = a + b(q_0 + 1)q_1 + c(q_0 + 1)q_0 q_1^2 \quad \text{for } n = 1,$$

while

$$(q_0 + 1)q_1(q_0 + 1) \prod_{j=1}^{2n-1} q_j = a(q_0 + 1) \prod_{j=1}^{2n-3} q_j + b(q_0 + 1) \prod_{j=1}^{2n-1} q_j + c(q_0 + 1) \prod_{j=1}^{2n+1} q_j$$

for $n > 1$. Arguing as above, we conclude that $c = 1$ for every n.

In order to determine b we argue as follows. The product of two representatives γ_2 in Γ_2 and γ_{2n} in Γ_{2n} lies in Γ_{2n}, if and only if γ_{2n} moves the vertex o by $2n$ units, γ_2 moves o by 2 units, and their product moves o by $2n$ units. This happens if and only if the segments $[o, \gamma_{2n} \cdot o]$ and $[o, \gamma_2^{-1} \cdot o]$ intersect in precisely one edge, which gives us $b = (q_1 - 1)$ possibilities for the representative γ_2, as claimed in statement (3)(b) of our theorem. Finally $a = (q_0 + \delta_{n,1})q_1$ follows from the above equation. $\quad \Box$

The multiplication table for the canonical basis of $\mathbb{C}[\Gamma, \Gamma_0]$ may be derived from the recursion relations by induction on the parameter n. We leave the details to the reader.

Corollary 3.5. *The complete multiplication table for the elements of the canonical basis of* $\mathbb{C}[\Gamma, \Gamma_0]$ *is given by*:

(1) *if Γ acts transitively on $\mathcal{T}^0$, then*

$$\Gamma_n \Gamma_m = \Gamma_m \Gamma_n = \Gamma_{m+n} + q^{n-1}(q + \delta_{mn})\Gamma_{m-n} + (q-1)\sum_{l=1}^{n-1} q^{l-1}\Gamma_{m+n-2l}$$

for $m \geqslant n > 0$;

(2) *if Γ acts with two orbits on $\mathcal{T}^0$, letting $q_l = q_0$ for l even and $q_l = q_1$ for l odd, then*

$$\Gamma_{2n} \Gamma_{2m} = \Gamma_{2m} \Gamma_{2n} = \Gamma_{2(m+n)} + q_1^n q_0^{n-1}(q_0 + \delta_{mn})\Gamma_{2(m-n)} + \sum_{l=1}^{2n-1}(q_l - 1)\prod_{i=1}^{l-1} q_i \Gamma_{2(m+n-l)}$$

for $m \geqslant n > 0$.

We close this section with several remarks.

Remark 3.6.

(1) It follows from part (1) of Theorem 3.3 that $\mathbb{C}[\Gamma, \Gamma_0]$ does not have an enveloping C^*-algebra because evaluation of polynomials at real numbers gives a set of $*$-representations of $\mathbb{C}[\Gamma, \Gamma_0] \cong \mathbb{C}[T]$, whose norms are not uniformly bounded.

(2) It is often convenient to work with 'normalized double cosets,' which are obtained by scaling a double coset with the inverse of the value that the R-function attains there, thus putting $\widetilde{\Gamma}_k := R(\Gamma_k)^{-1}\Gamma_k$. The recurrence relations for these normalized double cosets become

$$\widetilde{\Gamma}_1\widetilde{\Gamma}_n = \frac{1}{q+1}(\widetilde{\Gamma}_{n-1} + q\widetilde{\Gamma}_{n+1}), \quad n \geqslant 1;$$

$$\widetilde{\Gamma}_2\widetilde{\Gamma}_{2n} = \frac{1}{(q_0+1)q_1}\big(\widetilde{\Gamma}_{2n-2} + (q_1-1)\widetilde{\Gamma}_{2n} + q_0q_1\widetilde{\Gamma}_{2n+2}\big), \quad n \geqslant 1.$$

The first set of recurrence relations are those obtained for the spherical functions on the homogeneous tree $\mathcal{T}$ by Figà-Talamanca and Nebbia in [15], via the consideration of averaging operators on the homogeneous tree. The second set of relations corresponds to the averages obtained by Gerardin and Lai in [16] for the semi-homogeneous tree, although there is no specific reference to Hecke algebras; the relations appear explicitly in the appendix of Parkinson's thesis [26, Theorem C.1.4].

(3) Theorem 3.3 applies to the action of $\mathrm{SL}_2(\mathbb{Q}_p)$ on its Bruhat–Tits tree. For this action there is a vertex whose stabilizer is the subgroup $\mathrm{SL}_2(\mathbb{Z}_p)$. That the Hecke algebra of the pair $(\mathrm{SL}_2(\mathbb{Q}_p), \mathrm{SL}_2(\mathbb{Z}_p))$ does not have a universal C^*-algebra was observed by Hall in Proposition 2.21 of her PhD thesis [17].

4. Strongly transitive automorphism groups relative to the pointwise stabilizer of an edge

In this section we assume that $\mathcal{T}$ is a locally finite tree with a distinguished edge e. The theme of the section is the structure of the Hecke algebra of a given group of automorphisms of $\mathcal{T}$ with respect to the subgroup that fixes e. Motivated by the case of algebraic groups, we shall impose an extra transitivity assumption on the type-preserving automorphisms of the given group.

Definition 4.1. A type-preserving group of automorphisms of a tree is *strongly transitive* if it acts transitively on the set of doubly-infinite geodesics in the tree and if the stabilizer of some doubly-infinite geodesic $\mathcal{A}$ acts transitively on the set of edges in $\mathcal{A}$.

Although the current context includes groups that are not type-preserving, we have chosen to restrict the definition of strong transitivity to actions of type-preserving automorphisms, for consistency with the established literature. This restriction is natural in the context of algebraic groups because of the link with *BN*-pairs. In that context, releasing the type-preserving assumption would mean that the Weyl group would no longer be a Coxeter group in general with significant consequences for the last *BN*-pair condition. It will be convenient to list here some reformulations of the property of strong transitivity. The proof is left as an exercise for the reader.

Lemma 4.2. *Suppose that G is a group of type preserving automorphisms of a locally finite tree $\mathcal{T}$; then the following conditions are equivalent.*

(1) *G is strongly transitive.*
(2) *G acts transitively on the edge set $\mathcal{T}^1$ and there exists an edge $f \in \mathcal{T}^1$ whose stabilizer acts transitively on the set of all doubly-infinite geodesics through f.*
(3) *G acts transitively on the edge set $\mathcal{T}^1$ and the stabilizer of each edge acts transitively on the set of all doubly-infinite geodesics through that edge.*

(4) *For any two doubly-infinite geodesics $\mathcal{A}_1$ and $\mathcal{A}_2$ and any two edges $f_1 \in \mathcal{A}_1$ and $f_2 \in \mathcal{A}_2$, there exists an element $g \in G$ such that $g \cdot \mathcal{A}_1 = A_2$ and $g \cdot f_1 = f_2$.*

It follows from Lemma 4.2 that the stabilizer of an arbitrary doubly-infinite geodesic is transitive on its set of edges. Next we consider group automorphisms of a locally finite tree such that the subgroup of type-preserving automorphisms is strongly transitive. We will show that the stabilizer subgroup of the distinguished edge e is a Hecke subgroup and we will characterize the corresponding Hecke algebra. This context includes, for instance, the group of all automorphisms (respectively all type-preserving automorphisms) of a (semi-)homogeneous tree and also natural examples arising from semisimple matrix groups of rank 1 over local fields, for which the underlying trees are the associated Bruhat–Tits trees. Specifically, working along the lines of [19], we will show that when the subgroup G^+ of type preserving automorphisms is strongly transitive, $(G, \mathcal{B})$ and $(G^+, \mathcal{B})$ are Hecke pairs and we will derive the structure of their Hecke algebras.

Proposition 4.3. *Let G be a group acting by automorphisms of a tree $\mathcal{T}$ in such a way that the subgroup G^+ of type-preserving automorphisms is strongly transitive. For a distinguished edge e of $\mathcal{T}$, let $\mathcal{B}$ be the subgroup of G fixing e pointwise and let $\mathcal{A}$ be a doubly-infinite geodesic containing e. Then $\mathcal{T}$ is semi-homogeneous and the action of G on $\mathcal{T}^0$ is either transitive (iff $G \neq G^+$) or else has two orbits (iff $G = G^+$). If $G \neq G^+$ there is in G an inversion i of the edge e that stabilizes $\mathcal{A}$, yielding a short exact sequence $1 \to G^+ \to G \to \mathbb{Z}/2\mathbb{Z} \to 1$. This short exact sequence splits if and only if G contains an inversion that is an involution. Furthermore, both $(G, \mathcal{B})$ and $(G^+, \mathcal{B})$ are Hecke pairs.*

Proof. If $G = G^+$ everything except the statement about Hecke pairs follows easily from Lemma 4.2. Suppose now that $G \neq G^+$. Then there exists $g \in G$ which is not type-preserving. Using Lemma 4.2(4) we can then construct an element $i \in G$ which inverts the edge $e \in \mathcal{T}^1$ and which stabilizes the doubly-infinite geodesic $\mathcal{A}$ containing e. Then

$$G = G^+ \cup G^+ i \tag{4.1}$$

from which it follows that there exists a short exact sequence

$$1 \to G^+ \to G \to \mathbb{Z}/2\mathbb{Z} \to 1$$

that splits if and only if G contains an inversion that is an involution.

Next we verify that the pairs of groups $(G^+, \mathcal{B})$ and $(G, \mathcal{B})$ are both Hecke pairs. Let X be the metric space whose set of points consists of the midpoints of all edges of $\mathcal{T}$ (defined combinatorially as the set of unordered pairs $\{f, \overline{f}\}$, where f runs through all edges of $\mathcal{T}$) and whose distance function assigns the number n to the pair consisting of the midpoint of f_1 and the midpoint of f_2 whenever the longest geodesic path joining some vertex of f_1 to some vertex of f_2 has length $n + 1$.

The group G, and hence also its subgroup G^+, act by isometries on X. Denote the midpoint of the edge e by o. The stabilizer of o in G^+ (respectively in G) is $\mathcal{B}$ (respectively the normalizer $N_G(\mathcal{B})$ of $\mathcal{B}$ in G). Since $\mathcal{T}$ is locally finite, part (2.1) of Lemma 2.1 implies that $(G^+, \mathcal{B})$ and $(G, N_G(\mathcal{B}))$ are Hecke pairs. Since the index of $\mathcal{B}$ in $N_G(\mathcal{B})$ is at most 2, it follows that $(G, \mathcal{B})$ is a Hecke pair also in the case $G \neq G^+$. $\square$

Theorem 4.4. *Under the same conditions of hypothesis of Proposition* 4.3 *we have the following*:

(1) *The algebras* $\mathbb{C}[G, \mathcal{B}]$ *and* $\mathbb{C}[G^+, \mathcal{B}]$ *are isomorphic to the group algebra of the infinite dihedral group,* D_∞.

(2) *The canonical basis of the Hecke algebras* $\mathbb{C}[G^+, \mathcal{B}]$ *(respectively* $\mathbb{C}[G, \mathcal{B}]$ *if* $G \neq G^+$*) consisting of the double cosets with respect to* $\mathcal{B}$*, are parametrized by the automorphism group of the bipartite graph (respectively the abstract graph) underlying* $\mathcal{A}$*. More precisely*:

    (a) *Denote the stabilizer of the geodesic* $\mathcal{A}$ *in* G^+ *by* N^+ *and the pointwise stabilizer of* $\mathcal{A}$ *in* G^+ *by* T*. Then*

        (i) *every automorphism of* $\mathcal{A}$ *as a bipartite graph is realized by some element of* N^+ *and hence* N^+/T *is isomorphic to the group of automorphisms of the bipartite graph* $\mathcal{A}$*, the infinite dihedral group* D_∞;

        (ii) *we have* $T = N^+ \cap \mathcal{B}$*, and therefore, given an element n of* N^+ *with image w in the quotient group* $W := N^+/T$*, the left, right and double cosets of all elements in* nT *with respect to* $\mathcal{B}$ *are equal and we may consequently write them as* $w\mathcal{B}$*,* $\mathcal{B}w$ *and* $\mathcal{B}w\mathcal{B}$ *respectively*;

        (iii) *with the conventions justified in* (ii) *above, the group* G^+ *is the disjoint union* $\bigsqcup_{w \in W} \mathcal{B}w\mathcal{B}$.

    (b) *If* $G \neq G^+$*, denote the stabilizer of the geodesic* $\mathcal{A}$ *in* G *by* N *and the pointwise stabilizer of* $\mathcal{A}$ *in* G *by* T*. Then*

        (i) *every graph automorphism of* $\mathcal{A}$ *is realized by some element of* N *and* $\widetilde{W} := N/T$ *is isomorphic to the group of automorphisms of the graph* $\mathcal{A}$*, which is the split extension of* N^+/T *by the group* $N/N^+ \cong \mathbb{Z}/2\mathbb{Z}$*, and is abstractly isomorphic to the infinite dihedral group* D_∞;

        (ii) *since T is contained in* $\mathcal{B}$*, given an element $\widetilde{n}$ of* N *with image $\widetilde{w}$ in the quotient group* $\widetilde{W} := N/T$*, the left, right and double cosets of all elements in* $\widetilde{n}T$ *with respect to* $\mathcal{B}$ *are equal and we may consequently write them as* $\widetilde{w}\mathcal{B}$*,* $\mathcal{B}\widetilde{w}$ *and* $\mathcal{B}\widetilde{w}\mathcal{B}$ *respectively*;

        (iii) *with the conventions justified in* (ii) *above, the group* G *is the disjoint union* $\bigsqcup_{\widetilde{w} \in \widetilde{W}} \mathcal{B}\widetilde{w}\mathcal{B}$.

    Denote by s and t the generators of D_∞ that map the edge e to its two neighbors in $\mathcal{A}$*, the elements of* $\mathbb{C}[G^+, \mathcal{B}]$ *and* $\mathbb{C}[G, \mathcal{B}]$ *defined by* $\mathcal{B}w\mathcal{B}$ *for w in W and $\widetilde{W}$ by* Δ_w *and, if* $G \neq G^+$ *the element of* $\mathbb{C}[G, \mathcal{B}]$ *with underlying set* $\mathcal{B}i\mathcal{B} = \mathcal{B}i = i\mathcal{B}$ *by* Δ_i*. The image of the element i in the group $\widetilde{W}$ will also be denoted by i; it interchanges the odd and even vertex of the edge e and conjugates s to t.*

    The algebra $\mathbb{C}[G^+, \mathcal{B}]$ *is generated by the identity,* $\mathcal{B}$*,* Δ_s *and* Δ_t*, and in case* $G \neq G^+$*, the algebra* $\mathbb{C}[G, \mathcal{B}]$ *is generated by the identity,* $\mathcal{B}$*,* Δ_i *and* Δ_s*. More precisely: Suppose that w is an element of W. If* $s_1 \cdots s_n$ *is a reduced decomposition of* $w \in W$ *as a word in* $\{s, t\}$ *then* $\Delta_w = \Delta_{s_1} \cdots \Delta_{s_n}$ *and* $R(\Delta_w) = \prod_{i=1}^{n} R(\Delta_{s_i})$*. Furthermore, for every $\widetilde{w}$ in $\widetilde{W}$, we have* $\Delta_i \Delta_{\widetilde{w}} = \Delta_{i\widetilde{w}}$.

(3) (a) *Left and right multiplication with the generators* Δ_r*;* $r \in \{s, t\}$ *on the basis of the* $\mathbb{Z}$*-module underlying* $\mathbb{Z}[G^+, \mathcal{B}]$ *that consists of the elements* Δ_w*;* $w \in W$ *is given by*

$$\Delta_r \Delta_w = q_r \Delta_{rw} + (q_r - 1)\Delta_w \quad \text{if } w \text{ begins with } r,$$

$$\Delta_w \Delta_r = q_r \Delta_{wr} + (q_r - 1)\Delta_w \quad \text{if } w \text{ ends with } r,$$

$$\Delta_r \Delta_w = \Delta_{rw} \quad \text{if } w \text{ doesn't begin with } r,$$

$$\Delta_w \Delta_r = \Delta_{wr} \quad \text{if } w \text{ does not end with } r.$$

Furthermore, setting $q_r := R(\Delta_r)$, *the set of relations*

$$\Delta_r^2 - \big(q_r + (q_r - 1)\Delta_r\big) \quad \big(r \in \{s, t\}\big),$$

define a presentation of $\mathbb{Z}[G^+, \mathcal{B}]$ *as a ring with unit.*

(b) *Suppose that* $G \neq G^+$. *For* w *in* W *write* $\overline{w}$ *for the image of* w *under the automorphism of* W *which exchanges the generators* s *and* t; *this automorphism is implemented by conjugation with the element* iT *in the group* $\widetilde{W}$. *The group* $\mathbb{Z}/2\mathbb{Z}$ *acts on* $\mathbb{Z}[G^+, \mathcal{B}]$ *with the non-trivial element therein sending* Δ_w *to* $\Delta_{\overline{w}}$ *for* w *in* W. *The Hecke algebra* $\mathbb{Z}[G, \mathcal{B}]$ *is isomorphic to the twisted tensor product of this action of* $\mathbb{Z}/2\mathbb{Z}$ *on* $\mathbb{Z}[G^+, \mathcal{B}]$. *Furthermore, the set of relations*

$$\Delta_r^2 - \big(q_r + (q_r - 1)\Delta_r\big) \quad \big(r \in \{s, t\}\big),$$
$$\Delta_i^2 - 1,$$
$$\Delta_i \Delta_s \Delta_i^{-1} - \Delta_t$$

define a presentation of $\mathbb{Z}[G, \mathcal{B}]$ *as a ring with unit.*

Proof. We begin by verifying the statements about the canonical bases of the Hecke algebras $\mathbb{C}[G^+, \mathcal{B}]$ and $\mathbb{C}[G, \mathcal{B}]$ made in claim (2), starting with part (a) of that claim.

Since the group G^+ acts strongly transitively on $\mathcal{T}$, the group N^+ is transitive on the set of edges of the geodesic $\mathcal{A}$ by Lemma 4.2. This implies that every automorphism of $\mathcal{A}$ as a bipartite graph is realized by some element of N^+, showing the first claim made in part (2)(a)(i). Since T is the kernel of the action of N^+ on $\mathcal{A}$, this implies that N^+/T is isomorphic to the group of type-preserving automorphisms of a triangulated line. Since the latter group is the infinite dihedral group, D_∞, claim (2)(a)(i) follows.

Every element stabilizing a doubly-infinite geodesic and fixing an edge thereon must act trivially on that geodesic. This shows that $N^+ \cap \mathcal{B} \subseteq T$. Since the opposite inclusion is obvious, we have shown that $N^+ \cap \mathcal{B} = T$. In particular T is contained in $\mathcal{B}$, from which the remaining claims of statement (2)(a)(ii) follow immediately.

Let g be an element of G^+. By definition of $\mathcal{B}$, $\mathcal{B}g\mathcal{B}$ is the set of elements of G^+ which map the edge e to some element in the set $\mathcal{B}g.e$. Since the group N^+ is transitive on the set of edges of $\mathcal{A}$ and the group $\mathcal{B}$ is transitive on the set of doubly-infinite geodesics containing the edge e, every edge is contained in the set $\mathcal{B}N^+.e$. We conclude that $G^+ = \mathcal{B}N^+\mathcal{B}$.

To finish the proof of part (2)(a)(iii), suppose that w and w' are two elements of W such that $\mathcal{B}w\mathcal{B} = \mathcal{B}w'\mathcal{B}$. We have to show that w equals w'. For every element g of G^+, there is a unique edge of $\mathcal{A}$ contained in $\mathcal{B}N^+.e$. If g is an element of N^+ mapping to an element $\widehat{w}$ of W, that edge is $\widehat{w}.e$. By the definition of the group W, its action on the set of edges of $\mathcal{A}$ is simply transitive, and we conclude that w equals w' as claimed. This completes the proof of (2)(a).

We now turn to part (b) of claim (2) which deals with the parametrization of double cosets in the case $G \neq G^+$.

We have seen in part (2)(a)(i) that all automorphisms of the bipartite graph $\mathcal{A}$ are realized by an element of the subgroup N^+ of N. Furthermore, Proposition 4.3 guarantees the existence of an element i in N inverting the edge e. Therefore all graph automorphisms of $\mathcal{A}$ are realized

by some element of the group N, showing the first claim made in part (i) of claim (2)(b). Since T is the kernel of the action of N on $\mathcal{A}$, this implies that N/T is isomorphic to the group of automorphisms of a triangulated line. The remaining statements of part (2)(a)(i) are left to the reader.

We have seen in part (ii) of claim (2)(a), which we have already established, that $T = N^+ \cap \mathcal{B}$. In particular, the group T is contained in $\mathcal{B}$. The remaining claims of statement (2)(b)(ii) follow from that fact and part (2)(b)(ii) is completely verified.

An element g of G which is not contained in G^+ can be written uniquely as g_+i, where g_+ is in G^+. It follows that g is contained in $\mathcal{B}g_+i\mathcal{B} = \mathcal{B}g_+\mathcal{B}i \subseteq \mathcal{B}N_+\mathcal{B}i = \mathcal{B}N_+i\mathcal{B} \subseteq \mathcal{B}N\mathcal{B}$, and we conclude that $G = \mathcal{B}N\mathcal{B} = \mathcal{B}\widetilde{W}\mathcal{B}$. It remains to show that the union $\bigcup_{\widetilde{w}\in\widetilde{W}} \mathcal{B}\widetilde{w}\mathcal{B}$ is disjoint. Now each element in $\mathcal{B}\widetilde{w}\mathcal{B}$ with $\widetilde{w}$ in $\widetilde{W}$ is type-preserving as an automorphism of $\mathcal{T}$ if and only if $\widetilde{w}$ is type-preserving as an automorphism of $\mathcal{A}$. Assume now that $\widetilde{w}$ and $\widetilde{w}'$ in $\widetilde{W}$ are such that $\mathcal{B}\widetilde{w}\mathcal{B} = \mathcal{B}\widetilde{w}'\mathcal{B}$. If either of $\widetilde{w}$ and $\widetilde{w}'$ is type-preserving, so is the other, both elements are seen to be contained in the subgroup W of $\widetilde{W}$ and they are equal by part (2)(a)(iii), which we already established. So the only remaining case to consider is the one where both $\widetilde{w}$ and $\widetilde{w}'$ do not preserve types. In this case choose elements w and w' in W such that $\widetilde{w} = w.iT$ and $\widetilde{w}' = w'.iT$. Our assumption $\mathcal{B}\widetilde{w}\mathcal{B} = \mathcal{B}\widetilde{w}'\mathcal{B}$ then implies that $\mathcal{B}w\mathcal{B}i = \mathcal{B}w'\mathcal{B}i$, which is equivalent to $\mathcal{B}w\mathcal{B} = \mathcal{B}w'\mathcal{B}$. The latter condition implies $w = w'$ by part (2)(a)(iii) which we already established. We conclude that $\widetilde{w}$ equals $\widetilde{w}'$ in that case also. This shows that $\bigcup_{\widetilde{w}\in\widetilde{W}} \mathcal{B}\widetilde{w}\mathcal{B}$ is a disjoint union.

In order to finish the proof of claim (2) of Theorem 4.4 we need to establish the claims in its last paragraph. Only the statements about the factorization of the elements $\Delta_{\widetilde{w}}$ for $\widetilde{w}$ in $\widetilde{W}$ need to be proved, because the other statements follow from what we have already established so far.

We begin by proving the claims on Δ_w for w in W by induction on the length of the reduced decomposition of w as a word in the set of generators $\{s, t\}$. Both of these claims are clearly true if w is the identity of W. Assume the induction hypothesis for elements w' with reduced decomposition of length n and let w be an element of W with reduced decomposition $s_1 \cdots s_n s_{n+1}$ of length $n+1$.

The word $s_1 \cdots s_n$ is a reduced decomposition of $w' := s_1 \cdots s_n$. By the induction hypothesis, we have

$$\mathcal{B}s_1\mathcal{B}s_2\mathcal{B}\cdots\mathcal{B}s_n\mathcal{B} = \mathcal{B}w'\mathcal{B} \quad \text{and} \quad \prod_{i=1}^{n} R(\mathcal{B}s_i\mathcal{B}) = R(\mathcal{B}w'\mathcal{B}).$$

We claim that

$$\mathcal{B}w'\mathcal{B}s_{n+1}\mathcal{B} = \mathcal{B}w's_{n+1}\mathcal{B} \quad \text{and} \quad R(\mathcal{B}w's_{n+1}\mathcal{B}) = R(\mathcal{B}w'\mathcal{B})R(\mathcal{B}s_{n+1}\mathcal{B}). \tag{4.2}$$

Since R is a ring homomorphism from the Hecke algebra into the integers the statement needed for the inductive step will follow from these equations and the induction hypothesis and our claims involving the elements Δ_w will be proved.

Choose elements $n_{w'}$ and $n_{s_{n+1}}$ in N^+ mapping to w' and s_{n+1} respectively. To show the first equation listed in (4.2) above, it suffices to prove that $n_{w'}\mathcal{B}n_{s_{n+1}} \subseteq \mathcal{B}w's_{n+1}\mathcal{B}$, because this implies that $\mathcal{B}w'\mathcal{B}s_{n+1}\mathcal{B} \subseteq \mathcal{B}w's_{n+1}\mathcal{B}$; the opposite inclusion $\mathcal{B}w'\mathcal{B}s_{n+1}\mathcal{B} \supseteq \mathcal{B}w's_{n+1}\mathcal{B}$ is trivial. In order to verify the above inclusion we apply a product $n_{w'}bn_{s_{n+1}}$ with b in $\mathcal{B}$ to the edge e and find that it sends e to an edge with distance $n+1$ from e on the same side of e as the edges in $\mathcal{B}w's_{n+1}\mathcal{B}$. This is enough to imply the desired inclusion.

In order to establish the second equation listed in (4.2) above note that there are $R(\mathcal{B}s_{n+1}\mathcal{B})$ ways to extend a geodesic from e to $w'.e$ to a geodesic of length $n + 1$.

The statement involving the multiplication by the element Δ_i in the case $G \neq G^+$ follows from what we have just seen and $R(\Delta_i) = 1$. Also, since we have $\Delta_i \Delta_s \Delta_i = \Delta_t$, we can omit either Δ_s or Δ_t from the set $\{\mathcal{B}, \Delta_s, \Delta_t, \Delta_i\}$ of generators of $\mathbb{C}[G, \mathcal{B}]$. The proof of claim (2) of Theorem 4.4 is complete.

We now derive the structure of the Hecke algebra $\mathbb{C}[G^+, \mathcal{B}]$ as claimed in part (3)(a) of Theorem 4.4. We first verify the relations listed at the end of claim (3)(a).

For r in $\{s, t\}$ one verifies that $\mathcal{B}r\mathcal{B}r\mathcal{B} = \mathcal{B} \cup \mathcal{B}r\mathcal{B}$. Hence we have $\Delta_r^2 = \lambda + \mu \Delta_r$ with some positive integers λ, μ. Furthermore, λ is given by

$$\lambda = \left|\mathcal{B} \setminus \mathcal{B}r^{-1}\mathcal{B} \cap \mathcal{B}r\mathcal{B}\right| = |\mathcal{B} \setminus \mathcal{B}r\mathcal{B}| =: q_r,$$

and the value of μ is obtained by applying the homomorphism R to the equality $\Delta_r^2 = \lambda + \mu \Delta_r$; we get $q_r^2 = \lambda + \mu q_r$. We deduce $\mu = q_r - 1$, which proves the relation claimed.

That the relations give a defining set of relations can be proved in the same fashion as in the proof of Theorem 3.5 in [19]. The rest of claim (3)(a) is a consequence of what we have already seen. More precisely, the last two types of equations listed follow from the decomposition of Δ_w as a product in Δ_s and Δ_t corresponding to the reduced decomposition of w established in part (2)(a), which we already proved. The other types of equations follow using the following four observations: (i) under the conditions stated, the reduced decomposition of w begins/ends with r; (ii) using part (2)(a), we may write Δ_w as a product involving Δ_r at the beginning/end; (iii) we may then apply the relation in part (3)(a), which we have already verified; (iv) the desired equation then follows using part (2)(a) once more. This finishes the proof of claim (3)(a).

Before establishing claim (3)(b), we note that the isomorphism of the algebra $\mathbb{C}[G^+, \mathcal{B}]$ with $\mathbb{C}W \cong \mathbb{C}D_\infty$ can be completed following the outline in the second paragraph following the statement of Theorem 1.11 in [13], thus establishing the part of claim (1) concerned with $\mathbb{C}[G^+, \mathcal{B}]$.

We turn to the proof of part (3)(b). The structure of $\mathbb{C}[G, \mathcal{B}]$ can be described in terms of the structure of $\mathbb{C}[G^+, \mathcal{B}]$ using the reasoning of the analogous result, Proposition 3.8 of [19]. The description we obtain will be in terms of the concept of twisted tensor product that is introduced in the following definition.

Definition 4.5. Let Ω be a group acting by automorphisms on a ring R. Then the $\mathbb{Z}$-module $\mathbb{Z}[\Omega] \otimes_{\mathbb{Z}} R$ with the multiplication given on elementary tensors by

$$(\omega \otimes r) \cdot (\omega' \otimes r') := \omega \cdot \omega' \otimes {\omega'}^{-1}(r) \cdot r'$$

for ω, ω' in Ω and r, r' in R is called *the twisted tensor product* of the given action of Ω on R. The twisted tensor product of an action, α say, of a group Ω on R will be written $\mathbb{Z}[\Omega] \otimes_\alpha R$.

We return to the proof of part (3)(b) of Theorem 4.4.

The map sending the non-trivial element in $\mathbb{Z}/2\mathbb{Z}$ to Δ_i and the identity to Δ_e defines an injective ring homomorphism from $\mathbb{Z}[\mathbb{Z}/2\mathbb{Z}]$ into $\mathbb{Z}[G, \mathcal{B}]$ with image $\mathbb{Z}[N_G(\mathcal{B}), \mathcal{B}]$, since $\Delta_i \Delta_{\widetilde{w}} = \Delta_{i\widetilde{w}}$ for all $\widetilde{w}$ in $\widetilde{W}$ by claim (2) of Theorem 4.4, which we already proved. We will identify $\mathbb{Z}[\mathbb{Z}/2\mathbb{Z}]$ with its image under this homomorphism.

Part (2)(b)(iii) of Theorem 4.4 together with $\mathbb{Z}[\mathbb{Z}/2\mathbb{Z}] \cdot W = \widetilde{W}$ and $\mathbb{Z}[\mathbb{Z}/2\mathbb{Z}] \cap W = \{1\}$ identify $\mathbb{Z}[G, \mathcal{B}]$ as a $\mathbb{Z}$-module with the tensor product $\mathbb{Z}[N_G(\mathcal{B}), \mathcal{B}] \otimes_{\mathbb{Z}} \mathbb{Z}[G^+, \mathcal{B}]$ by the map $\rho \otimes \Delta_w \mapsto \Delta_\rho \Delta_w$ ($\rho \in \mathbb{Z}[\mathbb{Z}/2\mathbb{Z}]$, $w \in W$).

Now for every $\rho \in \mathbb{Z}/2\mathbb{Z}$, the element Δ_ρ is invertible with inverse $\Delta_{\rho^{-1}}$. Hence $\mathbb{Z}[\mathbb{Z}/2\mathbb{Z}]$ acts on $\mathbb{Z}[G^+, \mathcal{B}]$ through the setting $\rho(\Delta_w) := \Delta_\rho \Delta_w \Delta_{\rho^{-1}} = \Delta_{\rho w \rho^{-1}}$ ($\rho \in \mathbb{Z}[\mathbb{Z}/2\mathbb{Z}]$, $w \in W$). Also, we have that the non-trivial element of $\mathbb{Z}[\mathbb{Z}/2\mathbb{Z}]$ sends Δ_w to $\Delta_{\overline{w}}$ for w in W. Thus the multiplication law in the tensor product $\mathbb{Z}[\mathbb{Z}/2\mathbb{Z}] \otimes_{\mathbb{Z}} \mathbb{Z}[G^+, \mathcal{B}]$ is given by

$$(\rho \otimes \Delta_w) \cdot (\rho' \otimes \Delta_{w'}) = \rho \cdot \rho' \otimes \rho'^{-1}(\Delta_w) \cdot \Delta_{w'},$$

for ρ, ρ' in $\mathbb{Z}/2\mathbb{Z}$ and w, w' in W. Hence $\mathbb{Z}[G, \mathcal{B}]$ exhibits the announced structure of twisted tensor product. That the relations listed in part (3)(b) define a presentation of $\mathbb{Z}[G, \mathcal{B}]$ follows from this structure result and the presentation of $\mathbb{Z}[G, \mathcal{B}]$ listed in claim (3)(a). Likewise, the outstanding statement in claim (1) follows.

The proof of Theorem 4.4 is complete. $\quad\square$

Corollary 4.6. *The complete multiplication tables for the canonical generators are given by*:

(1) *For the algebra $\mathbb{C}[G^+, \mathcal{B}]$, let w and w' be in W. Denote by $w_{(i)}$ (respectively $w_{[i]}$) the word in $\{s, t\}$ consisting of (respectively missing) the last i letters of w (so that $w = w_{[i]} w_{(i)}$) and by $_{(i)}w'$ (respectively $_{[i]}w'$) the word consisting of (respectively missing) the first i letters of w' (so that $w = {}_{(i)}w' \, {}_{[i]}w'$); also, whenever r is in the set $\{s, t\}$, denote by u the other element of that set; for a word w with reduced decomposition $s_1 \cdots s_n$ in W put $q_w := \prod_{i=1}^n q_{s_i}$ and for i even respectively i odd let q_i equal q_r respectively q_u. With this notation, we have*
  - (a) *Suppose the last letter of the reduced decomposition of w differs from the first letter of the reduced decomposition of w'; then $\Delta_w \Delta_{w'} = \Delta_{ww'}$;*
  - (b) *Suppose now the last letter of the reduced decomposition of w coincides with the first letter of the reduced decomposition of w', let m be the length of the shorter word among w and w' and let s_i be the ith letter of w'; then $\Delta_w \Delta_{w'} = q_w \Delta_{w_{[m][m]}w'} + \sum_{i=0}^{m-1} q_{w_{(i)}}(q_i - 1)\Delta_{w_{[i]}s_{i[i]}w'}$.*

(2) *When $G \neq G^+$, the products of elements of the standard basis for the algebra $\mathbb{C}[G, \mathcal{B}]$ are determined using the information given in (4)(a) above and the following relations for w, w' in W: $\Delta_{iw}\Delta_{w'} = \Delta_i(\Delta_w\Delta_{w'})$, $\Delta_w\Delta_{iw'} = \Delta_i(\Delta_{\overline{w}}\Delta_{w'})$ and $\Delta_{iw}\Delta_{iw'} = \Delta_{\overline{w}}\Delta_{w'}$.*

Proof. Part (1) is by induction on the quantity m while part (2) follows from part (1) and the relations involving the element Δ_i that were obtained in parts (2) and (3)(b) of Theorem 4.4. $\quad\square$

We close this section with an example showing that the short exact sequence $1 \to G^+ \to G \to \mathbb{Z}/2\mathbb{Z} \to 1$ does not always split.

Example 4.7. Let K be a local field with additive valuation ν and let G be the group $\mathrm{GL}_2(K)$ acting in the natural way on the Bruhat–Tits tree of $\mathrm{SL}_2(K)$. We show that the group G does not contain an inversion that is an involution.

First, by transitivity of G on edges, it suffices to show that in G no inversion of a fixed chosen edge is an involution. An element of G that inverts the edge fixed by the Iwahori subgroup of G must be a monomial matrix of the form

$$\begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix} \quad \text{with } |\nu(b) - \nu(c)| = 1.$$

The square of a matrix of the above form is bc times the identity matrix, which, by the condition on the valuations of b and c, cannot be the identity.

5. Stabilizer of an end and HNN extensions

In this section we consider another geometrically defined class of Hecke pairs arising from groups acting on a locally finite tree, $\mathcal{T}$. The Hecke pairs considered will consist of the stabilizer, B, of an end of $\mathcal{T}$ and its subgroup, M_0, stabilizing a distinguished vertex. Again we will assume that the group action is 'highly transitive' in a sense to be explained shortly. The assumptions imposed will equip B with the structure of an HNN-extension relative to an endomorphism α of M_0 such that the index $|M_0 : \alpha(M_0)|$ is finite. (Recall that this HNN-extension can be defined as the group with set of generators M_0 and a letter a not in M_0, subject to the relations in M_0 and the additional relations $am_0a^{-1} = \alpha(m_0)$ for $m_0 \in M_0$.) We remark that conversely, every HNN-extension satisfying the latter condition defines such a tree action, whose associated HNN-extension is isomorphic to the given one and the tree action can be reconstructed from its associated HNN-extension on its 'minimal subtree.'

The following proposition fixes our assumptions and derives basic properties of the situation under consideration; we leave its proof to the reader.

Proposition 5.1. *Let $\mathcal{T}$ be a locally finite tree, ∞ an end of $\mathcal{T}$, B be a group of automorphisms of $\mathcal{T}$ that stabilize ∞. Suppose that some element of B, s say, acts by a hyperbolic isometry. Replacing s by another element if necessary, we may assume that ∞ is attracting for s and that s has the smallest translation length amongst elements of B.*

Denote by M the subset of elements of B that have a fixed point in $\mathcal{T}$. Then M is a normal subgroup of B with infinite cyclic quotient, generated by the image of s. We have a split exact sequence $1 \to M \to B \to \mathbb{Z} \to 1$; hence B is the semidirect product $M \rtimes \mathbb{Z}$, with either 1 or -1 in $\mathbb{Z}$ acting via conjugation by s on B.

Let o be a vertex on the axis of s and let M_0 be the stabilizer of o in B. Then $M_0 \subseteq M$ and $M = \bigcup_{n \in \mathbb{N}} s^n M_0 s^{-n}$. Hence B is isomorphic to the HNN-extension of the group M_0 with respect to the endomorphism of M_0 defined by conjugation with s^{-1}, which will be denoted α.

The index $|M_0 : \alpha(M_0)|$ is finite and at most equal to $|\{x \in \mathcal{T}^0 \colon d(x, o) = d(s^{-1} \cdot o, o)\}| - 1$. Therefore s^{-1} commensurates M_0, and hence so does $\langle s, M_0 \rangle = B$. Thus (B, M_0) and (M, M_0) are Hecke pairs.

In the following we will study the algebra $\mathbb{C}[B, M_0]$ for some choices of B. Again there are many examples of groups whose action on a tree satisfies the assumptions of Proposition 5.1: The group of all automorphisms of a locally finite (semi-)homogeneous tree that fix an end satisfies the assumptions on B above, as do the stabilizers of an end inside any semisimple matrix group of rank 1 over a local field with the group acting by automorphisms of its Bruhat–Tits tree.

In contrast to Sections 3 and 4 however, the degree of transitivity imposed by the assumptions of Proposition 5.1 is not strong enough to ensure that all the resulting Hecke-algebras are isomorphic; the cause for this phenomenon is that if 0 denotes the end of the axis of s different from ∞, then the group B does not necessarily act doubly-transitively on the set $B.0$.

In the following subsection we show that the algebra $\mathbb{C}[B, M_0]$ can always be described as the crossed product of the algebra $\mathbb{C}[M, M_0]$ by the endomorphism α. We also derive some general properties of the algebra $\mathbb{C}[M, M_0]$. However, we determine the complete structure of the algebra $\mathbb{C}[M, M_0]$ only for some special cases, see Sections 5.3 and 5.4 below.

5.1. Reduction to the centralizer of an end relative to a vertex stabilizer

In the context of Proposition 5.1 the group M is said to be the 'centralizer' of the end ∞ because it consists of the elements of B that fix some representative of the end ∞ pointwise (a property which e.g. the non-zero powers of the element s do not have). The description of the algebra $\mathbb{C}[B, M_0]$ in terms of the algebra $\mathbb{C}[M, M_0]$ that we promised above exhibits $\mathbb{C}[B, M_0]$ as the *-algebraic semigroup crossed product $\mathbb{C}[M, M_0] \rtimes_\alpha \mathbb{N}$ as a consequence of the results proved in [21] and [2]. We state the result needed in the special case considered here for the convenience of the reader after clarifying some terms used in the statement.

We briefly recall first the definition of a semigroup crossed product and refer the reader to [23, p. 422] for a detailed definition. Suppose A is a unital C^*-algebra and P is a semigroup with an action $\alpha : P \to \operatorname{End} A$ of P on A by (not necessarily unital) endomorphisms. The *crossed product* of A by this action of P is an algebra, denoted $A \rtimes_\alpha P$, which is universal and minimal with respect to the conditions that A embed in $A \rtimes_\alpha P$ via a unital homomorphism i_A, that P embed in the isometries of $A \rtimes_\alpha P$ via a semigroup homomorphism i_P, and that

$$i_A\big(\alpha_x(a)\big) = i_P(x) i_A(a) i_P(x)^* \quad \text{for } x \in P \text{ and } a \in A.$$

A similar definition applies in the category of unital *-algebras, yielding an *algebraic semigroup crossed product*. At the purely algebraic level, a crossed product is a twisted tensor product as defined in Definition 4.5.

Finally, recall that a C^*-algebra is *approximately finite* if and only if it can be written as an inductive limit of finite-dimensional C^*-algebras.

Theorem 5.2. *Let B be the HNN-extension of a group M_0 with respect to an endomorphism, α, of M_0 such that $|M_0 : \alpha(M_0)|$ is finite. Denote by B the HNN-extension defined by the endomorphism α of M_0, name the automorphism of B induced by α again α and denote the dilated group $\bigcup_{n \in \mathbb{N}} \alpha^{-n}(M_0) \subseteq B$ by M. Then (B, M_0) and (M, M_0) are Hecke pairs and there is an action $\widehat{\alpha}$ of $\mathbb{N}$ by injective endomorphisms of the Hecke algebra $\mathbb{C}[M, M_0]$ given for $A \in M_0 \setminus M / M_0$ by the formula*

$$\widehat{\alpha}(A) = \big|M_0 : \alpha(M_0)\big|^{-1} \sum_{B \in M_0 \setminus \alpha^{-1}(A) / M_0} B.$$

*The Hecke algebra $\mathbb{C}[B, M_0]$ is isomorphic (as a unital *-algebra) to the semigroup crossed product $\mathbb{C}[M, M_0] \rtimes_{\widehat{\alpha}} \mathbb{N}$ via a map that extends the canonical injection $\mathbb{C}[M, M_0] \hookrightarrow \mathbb{C}[B, M_0]$. Furthermore, $\mathbb{C}[M, M_0]$ is the union of a directed family of finite-dimensional *-subalgebras, so $C_u^*[M, M_0]$ exists and is approximately finite; the action $\widehat{\alpha}$ extends to the C^*-algebra level and*

$$C_u^*(B, M_0) \cong C_u^*(M, M_0) \rtimes_{\widehat{\alpha}} \mathbb{N}.$$

Proof. That (B, M_0) and (M, M_0) are Hecke pairs follows from Proposition 5.1 and the statement at the end of the introductory paragraph to Section 5. The stated description of the structure of $\mathbb{C}[B, M_0]$ in terms of $\mathbb{C}[M, M_0]$ and α follows from Theorem 1.9 of [21].

Since $M = \bigcup_{n \in \mathbb{N}} \alpha^{-n}(M_0)$, the algebra $\mathbb{C}[M, M_0]$ is the increasing union of the algebras $\mathbb{C}[\alpha^{-n}(M_0), M_0]$; the latter are finite-dimensional algebras because for each natural number n

the index $|\alpha^{-n}(M_0):M_0|$ equals $|\alpha^{-1}(M_0):M_0|^n$ and hence is finite. Hence $\mathbb{C}[M,M_0]$ is approximately finite. As each M_0-double-coset in M is contained in a finite-dimensional $*$-algebra, it has finite spectrum. This implies that the universal Hecke C^*-algebra $C_u^*(M,M_0)$ exists and is approximately finite. The final statement relating $C_u^*(B,M_0)$ and $C_u^*(M,M_0)$ follows from Theorem 1.11 of [21]. □

5.2. K-theory considerations

The framework determined by Proposition 5.1 is particularly suitable to explore the K-theory of the algebra $\mathbb{C}[B,M_0]$. Indeed, since the C^*-algebra of the Hecke pair (M,M_0) is approximately finite, it has trivial K_1 groups, and the Pimsner–Voiculescu six term exact sequence for crossed products (see [28]) yields

$$
\begin{array}{ccccc}
K_0(C^*(M,M_0)) & \xrightarrow{(\text{id}-\alpha)_*} & K_0(C^*(M,M_0)) & \xrightarrow{(i)_*} & K_0(C^*(M,M_0))\rtimes_\alpha \mathbb{N} \\
\uparrow & & & & \downarrow \\
K_1(C^*(M,M_0))\rtimes_\alpha \mathbb{N} & \xleftarrow{(i)_*} & 0 & \xleftarrow{(\text{id}-\alpha)_*} & 0.
\end{array}
\tag{5.1}
$$

Thus the K groups of the Hecke C^*-algebra of (B,M_0) are given by the cokernel and kernel of the map $(\text{id}-\alpha)_*: K_0(C^*(M,M_0)) \to K_0(C^*(M,M_0))$:

$$
K_0\big(C^*(B,M_0)\big) = \text{coker}(\text{id}-\alpha)_* \quad \text{and} \quad K_1\big(C^*(B,M_0)\big) = \ker(\text{id}-\alpha)_*.
$$

Recall that $M = \bigcup_{k\in\mathbb{N}} \alpha^{-k}(M_0)$ and let $H_n := \mathbb{C}[\alpha^{-k}(M_0),M_0]$; then each H_k is finite-dimensional and thus a direct sum of full matrix algebras $H_k = \bigoplus_{i=1}^{n_k} M_{d(k,i)}$. The system of inclusions $H_k \hookrightarrow H_{k+1}$ is encoded by a Bratteli diagram; see [8] or [14] for a definition of the latter.

The K-theory $K_0(H_k)$ depends only on the number n_i of full matrix summands:

$$
K_0(H_k) = \mathbb{Z}^{n_k}
$$

and since

$$
C^*(M,M_0) = \bigcup_{n\in\mathbb{N}} H_n,
$$

one has

$$
K_0\big(C^*(M,M_0)\big) = \varinjlim\big(\mathbb{Z}^{n_k}, B_k\big)
$$

where the map $\mathbb{Z}^{n_k} \to \mathbb{Z}^{n_{k+1}}$ is encoded by some integer matrix B_k for each $k \in \mathbb{N}$. We will content ourselves with this brief general introduction here and tackle the explicit calculation of the K-theory of specific examples in a future article.

Next we will determine the structure of $\mathbb{C}[B,M_0]$ in the special cases in which the stabilizer of the end ∞ is taken first in the full automorphism group of a locally finite homogeneous tree, Theorem 5.4, and then in an algebraic group acting in its Bruhat–Tits tree. We rely on Theorem 5.2 mainly for motivation and guidance, but we give explicit calculations in both cases.

5.3. Stabilizer of an end relative to the stabilizer of a vertex – full $\operatorname{Aut} \mathcal{T}$ case

In this subsection we look at the simplest instance of the general set up of Proposition 5.1, in which B consists of all the automorphisms of the regular tree $\mathcal{T}$ that fix the distinguished end ∞.

Theorem 5.4. *Let $\mathcal{T}$ be a locally finite homogeneous tree of degree $q+1$, with ∞ a distinguished end and o a distinguished vertex of $\mathcal{T}$. Denote by B the stabilizer of ∞, by M the centralizer of ∞ and by M_0 the stabilizer of the vertex o. Let $s \in B$ be a hyperbolic element of translation length 1 such that ∞ is attracting for s and the axis of s contains the vertex o. Let $[s] := M_0 s M_0$, viewed as an element of $\mathbb{C}[B, M_0]$. Then we have the following descriptions of $\mathbb{C}[B, M_0]$ and $\mathbb{C}[M, M_0]$.*

(1) *For each positive integer n define M_n as the set of elements in M that fix $s^n \cdot o$ but not $s^{n-1} \cdot o$. The group M is the disjoint union of the sets M_n where n is a non-negative integer and this partition of M is the partition into double cosets with respect to the subgroup M_0. Furthermore $M_n^{-1} = M_n$ for any non-negative integer n and $R(M_n) = L(M_n) = (q-1)q^{n-1}$ for positive n. The multiplication on $\mathbb{C}[B, M_0]$ and $\mathbb{C}[M, M_0]$ is determined by*

$$[s]^{*n}[s]^n = q^n \quad for\ n \geqslant 0, \tag{5.2}$$

$$[s]^n[s]^{*n} = \sum_{i=0}^{n} M_i \quad for\ n \geqslant 0, \tag{5.3}$$

$$M_m M_n = M_n M_m = (q-1)q^{n-1} M_m \quad for\ m > n > 0, \tag{5.4}$$

$$M_m^2 = (q-2)q^{m-1} M_m + (q-1)q^{m-1} \sum_{i=0}^{m-1} M_i \quad for\ m > 0. \tag{5.5}$$

(2) *Let $\mu := q^{-1/2}[s]$. Then μ is a non-unitary isometry. The algebra $\mathbb{C}[M, M_0]$ is generated by the range projections $\{\mu^n \mu^{*n}\colon n \in \mathbb{N}\}$ and, as such, is isomorphic to the algebra of eventually constant sequences of complex numbers; in particular, it is abelian. The algebra $\mathbb{C}[B, M_0]$ is generated as a *-algebra by μ and, as such, is isomorphic to the universal *-algebra generated by an isometry.*

Proof. We begin by proving part (1), which identifies the elements of the canonical basis of the algebra $\mathbb{C}[M, M_0]$ in terms of their action on the tree.

That M is the disjoint union $\bigsqcup_{n \in \mathbb{N}} M_n$ follows from our initial definition of M_0 and the definition of the sets M_n for $n > 0$. It is also clear from the definition that $M_n^{-1} = M_n$ for every $n > 1$ and this, in turn, will imply that $R(M_n) = L(M_n)$ for any n, once we know that each M_n is an M_0 double coset.

Since M_0 fixes all the vertices $s^m \cdot o$ for m a non-negative integer, it follows that $M_0 M_n M_0 \subseteq M_n$ for all non-negative integers n, and hence each M_n is a union of M_0 double cosets in M. To see that each M_n consists of a single double coset, it suffices to consider positive n. Suppose then that m and m' are both contained in M_n for n positive. Let $m_0 \in M_0$ be an element that maps $m \cdot o$ into $m' \cdot o$ (there is such an element m_0, because the path from $s^n \cdot o$ to $m \cdot o$ is of the same length as that from $s^n \cdot o$ to $m' \cdot o$, and both paths are disjoint from the path $s^n \cdot o$ to o). Then the

element $m_0' := m'^{-1} m_0 m$ fixes o and thus is in M_0. We conclude that $m = m_0^{-1} m' m_0' \in M_0 m' M_0$ and thus m and m' are in the same M_0 double coset, as claimed.

It remains to verify that $L(M_n) = (q - 1)q^{n-1}$ for positive n. An M_0 left coset inside M_n is uniquely determined by the image of o under any of its elements. To count the possible images of o under elements of M_n, observe that any vertex at distance n from the vertex $s^n \cdot o$ is a possibility, except those in the branch starting at $s^{n-1} \cdot o$. Thus, there are $(q - 1)$ possibilities for the image of $s^{n-1} \cdot o$, and then q possibilities for each of the following vertices $s^{n-2} \cdot o$, $s^{n-3} \cdot o$, up to o. Thus there are $(q - 1)q^{n-1}$ possible images for o under elements of M_n. The proof of part (1) is now complete. As an intermediate step towards the multiplication formulae, we will prove

$$M_0 \subseteq s^n M_0 s^{-n} = M_{s^n \cdot o} \quad \text{for } n \geqslant 0, \quad \text{and} \tag{5.6}$$

$$(M_0 s M_0)^n (M_0 s^{-1} M_0)^n = s^n M_0 s^{-n} \quad \text{for } n > 1. \tag{5.7}$$

We begin by proving (5.6). By definition, M_0 is the stabilizer M_o of o in M, so $s^n M_0 s^{-n} = s^n M_o s^{-n} = M_{s^n \cdot o}$. Since o lies on the axis of s and since the fixed end ∞ is attracting for s, every element of M that fixes o, also fixes $s^n \cdot o$ for $n \geqslant 0$. This implies that $M_0 = M_o \subseteq M_{s^n \cdot o} = s^n M_0 s^{-n}$, finishing the proof of (5.6). Claim (5.7) can now be derived by induction on n using claim (5.6). Since $R(s) = |M_0 / s^{-1} M_0 s| = q$ and since $L(s) = 1$ by (5.6) above with $n = 1$, we may apply [9, Theorem 1.4] and conclude that $\mu := q^{-1/2}[s]$ is an isometry, from which we see that $[s]^*[s] = q1$. An easy induction argument then proves (5.3). Using (5.7) and (5.6) we see that the element $[s]^n[s]^{*n}$ of the algebra $\mathbb{C}[B, M_0]$ is supported on the set $M_{s^n \cdot o}$. which decomposes as the disjoint union $\bigsqcup_{k=0}^n M_k$ by the definition of the M_k's. Since each M_k is an M_0 double coset, $[s]^n[s]^{*n}$ can be expressed as a linear combination $\sum_{i=0}^n k_i(n) M_i$, and from claim (2) we see that $k_i(n) > 0$ for all non-negative integers n and all $0 \leqslant i \leqslant n$.

This observation, when combined with the homomorphism R of $\mathbb{Z}[M, M_0]$ to $\mathbb{Z}$ suffices to compute the coefficients $k_i(n)$. Apply the algebra-homomorphism R to the equation

$$[s]^n[s]^{*n} = \sum_{i=0}^n k_i(n) M_i. \tag{5.8}$$

We have already proved that $R(M_i) = (q - 1)q^{i-1}$ that $R(s) = q$ and that $R(s^{-1}) = L(s) = 1$. Using this information when applying R to Eq. (5.8) we get

$$q^n = k_0(n) + (q - 1) \sum_{i=1}^n k_i(n) q^{i-1}. \tag{5.9}$$

If we calculate the sum on the right-hand side of Eq. (5.9) substituting the minimal possible value 1 for the unknown quantities $k_i(n)$ we obtain q^n as the result. Since all summands on the right-hand side of Eq. (5.9) are positive, none of the numbers $k_i(n)$ can be bigger than 1 and we derive that $k_i(n) = 1$ for all non-negative integers n and all $0 \leqslant i \leqslant n$. This finishes the proof of (5.3).

Since $M_n^{-1} = M_n$ for every $n \geqslant 0$, the double cosets are self-adjoint, and $\mathbb{C}[M, M_0]$ is abelian. The multiplication table in Eqs. (5.4) and (5.5) can now be obtained via a recursion using (5.2) and (5.3).

Applying basic linear algebra to the expressions $[s]^n[s]^{*n} = \sum_{i=0}^n M_i$ and recalling that of $\mu = q^{1/2}[s]$, we see that each M_k is a linear combination of range projections of powers of μ. Since the M_k for k in $\mathbb{N}$ form a basis of $\mathbb{C}[M, M_0]$, the first statement of part (1) follows. The isomorphism with the algebra C_c of eventually constant sequences is achieved as follows. Let $P_i := \mu^i \mu^{*i}$ for each $i \geqslant 0$ and note that the elements $Q_i := P_i - P_{i+1}$ are self adjoint and satisfy $Q_i Q_j = \delta_{i,j} Q_i$, and $Q_i P_j = 0$ if $i < j$ and 1 if $i \geqslant j$. Since the generators of C_c, $q_i = \delta_i$ and $p_j = \chi_{\{j,j+1,\ldots\}}$ have the same multiplication table, we map the sequence $(x_0, x_1, \ldots, x_N, x_N, \ldots) \in C_c$ to the combination

$$x_0 Q_0 + x_1 Q_1 + \cdots + x_N P_N = x_0 + (x_1 - x_0)P_1 + (x_2 - x_1)P_2 + \cdots + (x_N - x_{N-1})P_N.$$

In order to verify that the map is isometric, note that

$$
\begin{aligned}
&\left\| x_0 + (x_1 - x_0)P_1 + (x_2 - x_1)P_2 + \cdots + (x_N - x_{N-1})P_N \right\| \\
&= \max\left\{ |x_0|, \left| x_0 + \sum_{i=1}^N x_i - x_i - 1 \right| : 1 \leqslant n \leqslant N \right\} \\
&= \max\{ |x_n| : 0 \leqslant n \leqslant N \} \\
&= \left\| (x_0, x_1, \ldots, x_N, x_N, \ldots) \right\|_{C_c}.
\end{aligned}
$$

Since the algebra $\mathbb{C}[B, M_0]$ is generated by μ and $\mathbb{C}[M, M_0]$, the second statement in part (1) follows from the first and the universal property of the Toeplitz-algebra (see [12]). This finishes the proof of part (2) and of the theorem. $\square$

5.4. The centralizer of an end relative to a vertex stabilizer – algebraic group cases

In this subsection we derive some information on the Hecke algebra of the centralizer of an end of the Bruhat–Tits tree relative to a vertex stabilizer for semisimple matrix groups of rank 1 over a local field, using the structure of these groups. In these groups the subgroup M decomposes as a semidirect product $Z_0 \ltimes U$, and we have $M_0 U = M$. This can be used to identify the Hecke algebra $\mathbb{C}[M, M_0]$ with a subalgebra of $\mathbb{C}[U, U \cap M_0]$ and to explain the multiplication of $\mathbb{C}[M, M_0]$ in terms of the action of Z_0 on U, see Theorem 5.5. As this subsection is rather technical, those unfamiliar with the underlying general theory might find it helpful to keep in mind the special case $G = \mathrm{SL}_2(k)$ discussed below in Example 5.6. A similar result for matrix groups over algebraic number fields can be found in [24, Proposition 1.6(4)] where the structures denoted $\mathfrak{A}$ and A_θ are Hecke algebras, as explained in [24, p. 38] which act on a countably infinite product of trees.

Theorem 5.5. *Let G be a semisimple matrix group over a local field k and ∞ an end of the Bruhat–Tits tree $\mathcal{T}$ of G over k. The stabilizer B of ∞ is the group of k-rational points of a k-parabolic subgroup, $\mathbf{B}$, of G; let U be the group of k-rational points of the unipotent radical of $\mathbf{B}$ and Z the group of k-rational points of a maximal k-split torus contained in $\mathbf{B}$. Denote by M the centralizer of ∞, by M_0 the stabilizer in M (equivalently, in B) of a distinguished vertex, o, of $\mathcal{T}$. Put $U_0 := M_0 \cap U$. Denote by 0 the end different from ∞ fixed by Z. We have $U \subseteq M$, $B = Z \ltimes U$ and $M = (Z \cap M) \ltimes U$. The group $Z \cap M$ fixes the line connecting ∞ to 0*

and we have $Z \cap M \subseteq M_0$, and $M_0 = (Z \cap M) \ltimes U_0$, in particular $M_0 U = M$. Also, the group $Z_0 := Z \cap M$ coincides with the elements of Z that fix the point o.

The following statements hold:

(1) *For u in U we have* (*writing conjugation as an exponent on the left*):

$$M_0 u M_0 \cap U = M_0 u M_0 U_0 \cap U = {}^{M_0} u U_0,$$

$${}^{m_0} u U_0 = {}^{m'_0} u U_0 \quad \text{if and only if} \quad m_0 m_0'^{-1} \text{ fixes } u \cdot o.$$

(2) *Every element of the standard basis of $\mathbb{C}[M, M_0]$ can be represented as $M_0 u M_0$ with u in U and for u, u' in U we have $M_0 u M_0 = M_0 u' M_0$ if and only if $u' \in {}^{M_0} u U_0$; if U is abelian, we can replace the last condition by $u' \in {}^{Z_0} u U_0$.*

(3) *The map $\nu : \mathbb{C}[M, M_0] \to \mathbb{C}[U, U_0]$ defined for m in M by*

$$\nu(M_0 m M_0) := \sum_{h:\, U_0 \backslash (M_0 m M_0 \cap U)/U_0} U_0 h U_0$$

induces an isomorphism of $*$-algebras between $\mathbb{C}[M, M_0]$ and its image under ν.

(4) *If U is abelian, and u is in U, we have*

$$\nu(M_0 u M_0) = \sum_{h:\, {}^{Z_0} u U_0 / U_0} h U_0 \in \mathbb{C}[U/U_0].$$

Proof. The structural properties of end stabilizers in semisimple matrix groups of rank 1 over a local field, that are stated in the introductory paragraph can be found for example in Subsection 1.4 in [1].

The first displayed formula in claim (1) is proved using the fact that U is normal in M. This fact together with the definition of M_0 yields the second displayed formula of claim (1) and establishes the first claim.

In claim (2), that every element of the standard basis of $\mathbb{C}[M, M_0]$ can be represented as $M_0 u M_0$ with u in U follows from $M_0 U = M$, which is stated in the introductory paragraph of Theorem 5.5. The rest of claim (2) follows from claim (1) which we already proved, while the statement about the case where U is abelian follows from the equation $M_0 = Z_0 U_0$ that is obtained from the introductory paragraph of Theorem 5.5.

Claim (3) is essentially Proposition 6.4 in [20] (we only need that (M, M_0) is a Hecke pair, U is a subgroup of M such that $M = M_0 U$ and that $U_0 = U \cap M_0$). Concretely, setting $G := M$, $H := U$ and $S := M_0$ in [20] yields the corresponding claim for the Hecke-algebras over the integers. Tensoring with $\mathbb{C}$ we obtain the claim on the level of algebras over the field $\mathbb{C}$, and a direct calculation shows that ν commutes with the $*$-operation, thus completing the proof of (3).

Finally, claim (4) follows from claims (1) to (3) and the proof of Theorem 5.5 is complete. $\quad\square$

To give the reader some impression on the complexity saved by applying the map ν, we remark that the group U is either abelian or metabelian[1] and that Z acts via characters on the root subgroups whose product is U. As a further, more concrete, illustration, we now discuss the

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[1] The metabelian case only arises if the affine root system of G is of type $\widetilde{BC}_1$.

case of the group $SL_2(k)$, where k is a local field with ring of integers $\mathcal{O}$; the reader may prefer to specialize further and assume that $k = \mathbb{Q}_p$ and $\mathcal{O} = \mathbb{Z}_p$.

Example 5.6 (*Illustration of* $(B, M, M_0, U, U_0, Z, Z_0)$ *in the case of* $SL_2(k)$, *k a local field*). The groups B, M, M_0, U, U_0, Z and Z_0 depend on the choice of two ends and a vertex on the line connecting them in the tree $\mathcal{T}$. The ends and the vertex defined by the standard basis in the model of the Bruhat–Tits tree $\mathcal{T}$ of $SL_2(k)$ as outlined in [29] leads to the following list:

(1) B is the group of upper-triangular matrices in $SL_2(k)$;
(2) M is the subgroup of matrices of B, whose diagonal entries are contained in $\mathcal{O}$;
(3) M_0 is the group of upper-triangular matrices in $SL_2(\mathcal{O})$;
(4) U is the group of upper-unitriangular matrices in $SL_2(k)$ (so all diagonal entries 1);
(5) U_0 is the group of upper-unitriangular matrices in $SL_2(\mathcal{O})$ (so all diagonal entries 1);
(6) Z is the group of diagonal matrices in $SL_2(k)$;
(7) Z_0 is the group of diagonal matrices in $SL_2(\mathcal{O})$.

The group U is isomorphic to the additive group of k and U_0 is isomorphic to the additive group of $\mathcal{O}$; hence $\mathbb{C}[U, U_0] \cong \mathbb{C}[k/\mathcal{O}]$. Also, Z is isomorphic to the multiplicative group of non-zero elements in k, while Z_0 is isomorphic to the group of units of $\mathcal{O}$; conjugation by an element $\lambda \in k^\times \cong Z$ maps an element $u \in k \cong U$ to $\lambda^2 u$. Hence, in this case, we have:

$$\nu \left(M_0 \begin{bmatrix} 1 & u \\ 0 & 0 \end{bmatrix} M_0 \right) = \sum_{h:\ (\mathcal{O}^\times)^2 u + \mathcal{O}/\mathcal{O}} h\mathcal{O} \in \mathbb{C}[k/\mathcal{O}].$$

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