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When does QP yield the exact solution to constrained NMPC?

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It is well known that the optimal control sequence for a linear system with a quadratic cost and linear inequality constraints over a finite optimisation horizon can be computed by means of a quadratic programme (QP). The aim of this article is to investigate when the optimal control sequence for a *non-linear* single-input single-output system also can be computed via QP. Our main contribution is to show that the optimal control sequence for non-linear systems, with a quadratic cost and linear inequality constraints can be computed in exact form via QP provided the optimisation horizon is no larger than a critical quantity that we name the 'input–output linear horizon'. The results do not require any linearisation technique and are applicable to general non-linear systems, provided their input–output linear horizon is larger than their relative degree.

Keywords: quadratic programme; non-linear systems; model predictive control; constraints; relative degree; input-output linear horizon

1. Introduction

Much of the success of control engineering in the algorithmic/computational arena is related to the desirable properties of quadratic programming. From a mathematical point of view, a quadratic programme (QP) of the type arising in control applications typically has a positive-semidefinite Hessian, and hence it has the very attractive feature of being a *convex* optimisation problem (and, under mild reasonable assumptions, a strictly convex problem). From an algorithmic point of view, a number of highly efficient numerical methods are available for solving QPs; for example, active set methods and interior point methods. From a physical viewpoint, it is often the case that the 'energy' of some variables of interest is to be minimised while satisfying some linear constraints (for example, hard bounds on their values). Moreover, quadratic cost functionals weight deviations around a target value equally irrespective of their sign, giving practically no importance to small deviations while heavily weighting large deviations away from the target.

Model predictive control (MPC) is a particular control technique that, in its most typical formulation, considers a quadratic cost functional to be minimised. MPC algorithms solve, at each sampling instant, the minimisation of the quadratic functional over a fixed window (time horizon) of the problem data and decision variables. When the underlying dynamical system is linear and the variables of interest are required to satisfy linear inequality constraints, MPC formulations with quadratic functionals can be cast as a QP, possessing all the aforementioned computational and theoretical advantages. On the other hand, when the underlying dynamical system is *non-linear*, the resulting optimisation problem, even with a quadratic cost and linear inequality constraints, becomes—in general—a non-linear programme (NLP) since the variables satisfying the system dynamics are related by non-linear equality constraints. Thus, many of the computational and theoretical advantages of a QP are lost as, in general, finding a global optimum for a NLP (if it exists) is a very difficult and computationally demanding task. There are numerous approaches that try to circumvent these difficulties resorting to Taylor (*approximate*) linearisation-based techniques. A methodology based on on-line linearisation and iterative calculations can be found in De Keyser (1998) and Blet, Megias, Serrano and de Prada (2002).

The aim of this article is to investigate when the underlying fixed-horizon optimisation problem solved in MPC for a *non-linear* system can also be computed, in exact form, via QP. Previously reported work in the literature addresses related issues and we give a brief overview below. For brevity of exposition, we will

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roughly classify previous work into two categories. The first category concerns approaches that linearise the system dynamics by state and control input transformation (feedback linearisation) and then formulate an MPC problem in terms of the new 'linearised' variables (see, e.g. Nevistić and Morari 1995). In this case, the non-linear problem is completely converted into a linear MPC problem and QP can be used to compute the solution. However, it has been shown that this 'lack of respect' for the MPC cost in terms of the original variables can result in arbitrarily poor performance (Primbs and Nevistić 1997). See also Freeman and Kokotovic (1995) for further discussion on the potential drawbacks of using feedback linearisation in the context of optimal control.

The second category of existing work attempting to solve non-linear MPC (NMPC) problems via QP concerns approaches that formulate MPC in terms of the original variables but linearise the system dynamics via a change of coordinates to simplify computation. This strategy is discussed, for example, in Primbs and Nevistić (1997). The advantage of this approach is that the relationships between the variables of the problem become linear, which reduces complexity in computing the predicted system trajectories over the horizon. One drawback is that the transformation of coordinates, especially in the control variable, transforms the cost functional into a non-quadratic and, in general, non-convex function. Some approaches have been reported that avoid this problem by not weighting the control effort in the cost functional (e.g. Bloemen, Cannon and Kouvaritakis 2002; Liao, Cannon and Kouvaritakis 2005) on the grounds that input constraints are explicitly accounted for.

Another problem, that both categories of existing work discussed above have, arises in the presence of constraints on the original control input. In this case, the non-linear transformation of the control input variable renders the constraints on the transformed input non-linear. Hence, exact constraint mapping requires computationally intensive non-linear programming (Nevistić and Del Re 1994) or iterative solution methods (Oliveira, Nevistić and Morari 1995), defeating the purpose of posing the original problem as a QP. Some remedies to this problem have been attempted in, for example, Kurtz and Henson (1998), where an *approximation* of the constraints is proposed. In contrast to the latter work, our aim here is to investigate when QP yields the *exact* solution to the original MPC problem.

The main contribution of the present article is to show that the optimal control sequence for non-linear single-input single-output systems, with a quadratic cost and linear inequality constraints can be computed in exact form via QP provided the optimisation horizon is no larger than a critical horizon. This critical horizon, over which MPC for non-linear systems can be solved via QP, is related to a novel concept introduced in this article, which we name the *input–output linear horizon* (IOLH). The IOLH is the number of sampling times before the input of a system appears on the output in *non-linear* form. The IOLH is clearly not less than the relative degree of the output and, depending on the structure of the non-linear system, it can take values up to two times the order of the system. Some properties of the IOLH are explored, including its relationship with the relative degree and invariance under coordinate transformation. The remainder of this article is organised as follows. In § 2, properties of the relative degree of a system output are reviewed, the IOLH is defined and some of its properties are explored. In § 3, it is shown how the IOLH can be used to solve an MPC problem formulated in the original input and output variables in exact form via QP. In § 4, we illustrate the use of QP to solve MPC for a non-linear system example consisting of a flexible joint manipulator. Finally, § 5 contains concluding remarks.

2. Relative degree and IOLH

Consider the following discrete-time single-input single-output non-linear dynamic system

$$x_{k+1} = F(x_k, u_k), \quad F\colon \mathbb{M} \times \mathbb{R} \to \mathbb{M}, \tag{1}$$

$$y_k = H(x_k), \quad H\colon \mathbb{M} \to \mathbb{R}, \tag{2}$$

where $x_k \in \mathbb{M}$, $u_k \in \mathbb{R}$ and $y_k \in \mathbb{R}$ denote the state, input and output, respectively, at time k, $\mathbb{M}$ is an open and dense subset of $\mathbb{R}^n$, and F and H are analytic functions on their domains.

Denoting by $\mathcal{U}_k := (u_i)_{i=0}^k = (u_0, \ldots, u_k)$, we use the following notation:

$$\begin{aligned} F_0(x, \mathcal{U}_0) &:= F(x, u_0), \\ F_{k+1}(x, \mathcal{U}_{k+1}) &:= F(F_k(x, \mathcal{U}_k), u_{k+1}), \quad k \geq 0, \end{aligned}$$

where $\mathcal{U}_{k+1} = (u_0, \ldots, u_{k+1})$. Hereafter, $\circ$ denotes composition of functions and $\mathbb{Z}_+$ denotes the set of strictly positive integers.

Definition 2.1 (Relative degree)**:** The discrete-time system (1) and (2) is said to have relative degree $r \in \mathbb{Z}_+$ in the set $\mathbb{M}$ if

(i) $(\partial/\partial u_0) H \circ F_k(x, \mathcal{U}_k) = 0$
$\forall\ (x, \mathcal{U}_k) \in \mathbb{M} \times \mathbb{R}^{k+1},\ 0 \leq k < r-1,$

(ii) $(\partial/\partial u_0) H \circ F_{r-1}(x, \mathcal{U}_{r-1}) \neq 0$ a.e. in $\mathbb{M} \times \mathbb{R}^r$.

It is said to have relative degree ∞ in $\mathbb{M}$ if (i) holds for all $k \geq 0$.

Note that $H \circ F_k(x, \mathcal{U}_k)$ is the output y_{k+1} when the input sequence $(u_i)_{i=0}^k = (u_0, \ldots, u_k)$ is applied to the system (1)–(2) with initial state $x_0 = x$. If a discrete-time system (1)–(2) has relative degree r, then it follows from the definition that y_r is the first sample of the output that depends on the control input u_0. It also follows, since the system is causal and time invariant, that y_r is the first sample of the output that depends on any input in the sequence $(u_i)_{i=0}^{\infty}$.

Note as well that Definition 2.1 is slightly different from the standard definition of relative degree in the literature (see, e.g. Monaco and Normand-Cyrot (1987)). Indeed, the latter reference uses a sequence of the form $(u_i)_{i=0}^k = (u_0, 0, \ldots, 0)$ in the definition. The following result establishes the equivalence.

Proposition 2.2: *The discrete-time system* (1)–(2) *has relative degree* (*in the sense of Definition* 2.1) $r \in \mathbb{Z}_+$ *in the set* $\mathbb{M}$ *if and only if* $(\partial/\partial u_0) H \circ F_k(x, (u_0, 0, \ldots, 0)) = 0 \;\; \forall\, (x, u_0) \in \mathbb{M} \times \mathbb{R},\; 0 \leq k < r - 1$, *and*

$$(\partial/\partial u_0) H \circ F_{r-1}(x, (u_0, 0, \ldots, 0)) \neq 0 \text{ a.e. in } \mathbb{M} \times \mathbb{R}.$$

Proof: The result follows from causality and time invariance of the model. □

If the relative degree is infinite then the output is unaffected by the control input $(u_i)_{i=0}^{\infty}$. Moreover, if the relative degree is finite then it must satisfy $1 \leq r \leq n$ (see, e.g. Monaco and Normand-Cyrot (1987)).

Definition 2.3 (Coordinate transformation)**:** $z = f(x)$, where $f: \mathbb{M} \to \mathbb{M}$ is an analytic invertible function, defines a coordinate transformation.

Definition 2.4 (Feedback)**:** $u = \gamma(x, v)$, where $\gamma: \mathbb{M} \times \mathbb{R} \to \mathbb{R}$ is an analytic function such that

$$(\partial/\partial v)\, \gamma(x, v) \neq 0 \text{ a.e. in } \mathbb{M} \times \mathbb{R},$$

defines a regular feedback law.

The following notation for the dynamics driven with feedback is introduced:

$$\bar{F}(x, v) := F(x, \gamma(x, v)). \tag{3}$$

As before, we define $\mathcal{V}_k := (v_i)_{i=0}^k = (v_0, \ldots, v_k)$ and use the following notation:

$$\bar{F}_0(x, \mathcal{V}_0) := \bar{F}(x, v_0),$$

$$\bar{F}_{k+1}(x, \mathcal{V}_{k+1}) := \bar{F}(\bar{F}_k(x, \mathcal{V}_k), v_{k+1}), \quad k \geq 0,$$

where $\mathcal{V}_{k+1} = (v_0, \ldots, v_{k+1})$.

The proof of the following proposition goes along the lines of the proof of Proposition 2.7 in Monaco and Normand-Cyrot (1995), but since the definition of relative degree presented here and the assumptions for the proposition are slightly different from the ones in the aforementioned reference, it is included for completeness.

Proposition 2.5: *Given the discrete-time system* (1)–(2), *its relative degree is invariant under coordinate transformation and feedback.*

Proof: Under the coordinate transformation $z_k = f(x_k)$, the system (1)–(2) can be written as

$$z_{k+1} = \tilde{F}(z_k, u_k), \tag{4}$$

$$y_k = \tilde{H}(z_k), \tag{5}$$

where $\tilde{F}(z_k, u_k) := f \circ F(f^{-1}(z_k), u_k)$ and $\tilde{H}(z_k) := H \circ f^{-1}(z_k)$. As before, we define $\tilde{F}_0(z, \mathcal{U}_0) := \tilde{F}(z, u_0)$ and $\tilde{F}_{k+1}(z, \mathcal{U}_{k+1}) := \tilde{F}(\tilde{F}_k(z, \mathcal{U}_k), u_{k+1})$, for $k \geq 0$.

To prove invariance under coordinate transformation it is sufficient to prove that

$$\tilde{H} \circ \tilde{F}_k(z, \mathcal{U}_k) = H \circ F_k(x, \mathcal{U}_k), \quad k \geq 0,\; x = f^{-1}(z). \tag{6}$$

This, along with Definition 2.1, will show that system (1)–(2) has relative degree r if and only if system (4)–(5) does.

First, we prove by induction that

$$\tilde{F}_k(z, \mathcal{U}_k) = f \circ F_k(x, \mathcal{U}_k), \quad k \geq 0,\; x = f^{-1}(z). \tag{7}$$

Note that (7) holds for $k = 0$ from the definitions of F_0 and $\tilde{F}_0$. Now assume that (7) holds for some $k \geq 0$, then it is easy to see that it holds for $k+1$ since

$$\begin{aligned} \tilde{F}_{k+1}(z, \mathcal{U}_{k+1}) &= \tilde{F}(\tilde{F}_k(z, \mathcal{U}_k), u_{k+1}) \\ &= \tilde{F}(f \circ F_k(x, \mathcal{U}_k), u_{k+1}) \\ &= f \circ F(F_k(x, \mathcal{U}_k), u_{k+1}) \\ &= f \circ F_{k+1}(x, \mathcal{U}_{k+1}). \end{aligned}$$

Then (7) holds for all $k \geq 0$. From (7) and the definitions of $\tilde{H}$ and $\tilde{F}$ it follows that:

$$\tilde{H} \circ \tilde{F}_k(z, \mathcal{U}_k) = H \circ f^{-1} \circ f \circ F_k(x, \mathcal{U}_k) = H \circ F_k(x, \mathcal{U}_k).$$

This completes the proof of invariance under coordinate transformation.

For invariance under feedback $u_k = \gamma(x_k, v_k)$, using (3) we have to prove that, if r is the relative degree of system (1) and (2), then:

$$\text{(i)} \quad \frac{\partial}{\partial v_0} H \circ \bar{F}_k(x, \mathcal{V}_k) = 0 \quad \forall\, (x, \mathcal{V}_k) \in \mathbb{M} \times \mathbb{R}^{k+1}, \quad 0 \leq k < r - 1,$$

(ii) $\frac{\partial}{\partial v_0} H \circ \bar{F}_{r-1}(x, \mathcal{V}_{r-1}) \neq 0$ a.e. in $\mathbb{M} \times \mathbb{R}^r$.

We prove first by induction that

$$\bar{F}_k(x, \mathcal{V}_k) = F_k(x, \mathcal{U}_k), \quad k \geq 0, \tag{8}$$

with $\mathcal{U}_k$ and $\mathcal{V}_k$ component-wise related by the feedback law $u_0 = \gamma(x, v_0)$, $u_k = \gamma(\bar{F}_{k-1}(x, \mathcal{V}_{k-1}), v_k), k \geq 1$. For $k = 0$, (8) is verified from the definitions of F_0 and $\bar{F}_0$. Now we assume that (8) holds for some $k \geq 0$, and we prove that this implies that (8) also holds for $k+1$:

$$\begin{aligned} F_{k+1}(x, \mathcal{U}_{k+1}) &= F(F_k(x, \mathcal{U}_k), u_{k+1}) = F(\bar{F}_k(x, \mathcal{V}_k), u_{k+1}) \\ &= F(\bar{F}_k(x, \mathcal{V}_k), \gamma(\bar{F}_k(x, \mathcal{V}_k), v_{k+1})) \\ &= \bar{F}(\bar{F}_k(x, \mathcal{V}_k), v_{k+1}) = \bar{F}_{k+1}(x, \mathcal{V}_{k+1}). \end{aligned}$$

We then have that

$$\begin{aligned} \frac{\partial}{\partial v_0} H \circ \bar{F}_k(x, \mathcal{V}_k) &= \frac{\partial}{\partial v_0} H \circ F_k(x, \mathcal{U}_k) \\ &= \frac{\partial}{\partial u_0} H \circ F_k(x, \mathcal{U}_k) \cdot \frac{\partial}{\partial v_0} \gamma(x, v_0) \end{aligned} \tag{9}$$

Since system (1) and (2) has relative degree r, the derivative with respect to u_0 in the right-hand side of (9) is 0 for $0 \leq k < r-1$, showing that condition (i) holds.

For $k = r-1$, since $(\partial/\partial v_0)\gamma(x, v_0) \neq 0$ a.e. in $\mathbb{M} \times \mathbb{R}$ by Definition 2.4, and from the definition of relative degree, it follows that the derivative with respect to u_0 in the right-hand side of (9) is different from 0 a.e. in $\mathbb{M} \times \mathbb{R}^r$. Thus, condition (ii) holds. This concludes the proof that the relative degree is invariant under feedback. □

We next introduce the IOLH. The IOLH is the number of sampling times before the input of a system appears on the output in *non-linear* form and will be a key element in answering the question posed in the title of this article.

Definition 2.6 (IOLH)**:** An analytic function $\zeta(x, u)$: $\mathbb{M} \times \mathbb{R} \to \mathbb{R}$ is said to have IOLH 0 in the set $\mathbb{M}$ under the dynamics of system (1) if $(\partial^2/\partial v_0{}^2)\zeta(x, u_0) \neq 0$ a.e. in $\mathbb{M} \times \mathbb{R}$. Otherwise, the function $\zeta(x, u)$ is said to have IOLH $\ell \in \mathbb{Z}_+$ in the set $\mathbb{M}$ under the dynamics of system (1) if

(i) $(\partial^2/\partial u_0 \partial u_i)\zeta(F_k(x, \mathcal{U}_k), u_{k+1}) = 0 \quad \forall\ (x, \mathcal{U}_{k+1}) \in \mathbb{M} \times \mathbb{R}^{k+2}, \quad 0 \leq k < \ell - 1, \quad 0 \leq i \leq k+1,$

(ii) $(\partial^2/\partial u_0 \partial u_i)\zeta(F_{\ell-1}(x, \mathcal{U}_{\ell-1}), u_\ell) \neq 0$ a.e. in $\mathbb{M} \times \mathbb{R}^{\ell+1}$ for some $0 \leq i \leq \ell$.

The function $\zeta(x, u)$ is said to have IOLH ∞ in $\mathbb{M}$ under the dynamics of system (1) if (i) holds for all $k \geq 0$. We denote the IOLH of a function ζ under the dynamics of system (1) by $\ell_{\zeta,F}$.

Of particular interest is the IOLH[1] $\ell_{H,F}$, which represents the number of samples it takes the input $u_0 = u$ to appear non-linearly in the output of system (1) and (2).

Remark 2.7: Note that Definition 2.6 of IOLH provides an extension of the concept of relative degree given in Definition 2.1 that uses input sequences of the general form $\mathcal{U}_k = (u_0, \ldots, u_k)$. Such extension does not work with the standard definition of relative degree (as in, e.g. Monaco and Normand–Cyrot (1987)) that uses $(u_0, 0, \ldots, 0)$ as input sequence, since the analogue of Proposition 2.2 does not hold for the definition of IOLH $\ell_{H,F}$. (Example 2.13 below illustrates this fact.)

Proposition 2.8: *Given the discrete-time system* (1)–(2), *its IOLH* $\ell_{H,F}$ *is invariant under coordinate transformation but, in general, not under feedback.*

Proof: The proof of invariance under coordinate transformation is analogous to the proof of invariance under coordinate transformation for the relative degree, and follows from (6) and Definition 2.6.

To show that the IOLH is, in general, not invariant under feedback, consider the example $x_{k+1} = \sin x_k + u_k$, $y_k = x_k$, for which $\ell_{H,F} = 2$. Under the feedback law $u_k = -\sin x_k + v_k$, the resulting system $x_{k+1} = v_k$, $y_k = x_k$, is linear. Therefore, the IOLH is now[2] $\ell_{H,\bar{F}} = \infty$. □

In a similar fashion as for the IOLH, one can define the input-state linear horizon (ISLH) as the number of samples it takes the input to appear non-linearly on any component of the state variable.

Definition 2.9 (ISLH)**:** Consider the discrete-time system (1) and output functions $H_i(x) := c_i\ x$, where c_i, $i = 1, \ldots, n$, are row vectors given by the ith row of the $n \times n$ identity matrix. Then, the ISLH $\ell_F \in \mathbb{Z}_+$ in the set M is defined as:

$$\ell_F := \min_{1 \leq i \leq n} \{\ell_{H_i,F}\}. \tag{10}$$

Proposition 2.10: *Given the discrete-time system* (1), *its ISLH* ℓ_F *is invariant under coordinate transformation but*, *in general*, *not under feedback*.

Proof: This result is a direct consequence of Definition 2.9 and Proposition 2.8. □

2.1 *Illustrative examples*

Given the discrete-time system (1) and (2) with relative degree r, it follows from Definitions 2.1 and 2.6 that the IOLH $\ell_{H,F} \geq r$. The following example illustrates a non-linear system structure with relative degree $r = n$ for which the IOLH can take values from $r = n$ to $2n$, depending on an index j.

Example 2.11: Consider the following family of non-linear systems (parameterised by the index j) with relative degree $r = n$:

$$\begin{aligned} x_{1,k+1} &= x_{2,k} \\ x_{2,k+1} &= x_{3,k} \\ &\vdots \\ x_{n-1,k+1} &= x_{n,k} \\ x_{n,k+1} &= \alpha(x_{j,k}) + \beta(u_k) \\ y_k &= H(x_k) = x_{1,k} \end{aligned}$$

where $x_k := (x_{1,k}, \ldots, x_{n,k})$, $1 \leq j \leq n$, $(\partial^2\alpha/\partial x_{j,k}^2) \neq 0$, and $(\partial\beta/\partial u_k) \neq 0$. Note that if $(\partial^2\beta/\partial u_k^2) \neq 0$, IOLH $\ell_{H,F} = n$. If $\beta(u_k) = Bu_k$, $B \in \mathbb{R}\backslash\{0\}$, then it follows by direct computation that applying the input sequence $\{u_0, u_1, u_2, \ldots\}$ the output sample $y_{2n-j+1} = \alpha(\alpha(x_j, 0) + Bu_0) + \mathrm{B}u_{n-j+1}$ is the first sample that shows a non-linear dependency on the input u_0. Thus, $\ell_{H,F} = 2n - j + 1 = r + n - j + 1$ and, as j can take values between 1 and $r = n$, the IOLH can take any value between $n + 1$ and $2n$.

The following example illustrates a structure of non-linear systems with relative degree less than n but otherwise arbitrary, where $\ell_{H,F} = 2n$.

Example 2.12: Consider the following non-linear system with arbitrary relative degree $1 \leq r \leq n-1$:

$$\begin{aligned} x_{1,k+1} &= x_{2,k} \\ x_{2,k+1} &= x_{3,k} \\ &\vdots \\ x_{r,k+1} &= Bu_k + x_{r+1,k} \\ x_{r+1,k+1} &= x_{r+2,k} \\ x_{r+2,k+1} &= x_{r+3,k} \\ &\vdots \\ x_{n,k+1} &= \alpha(x_{r+1,k}) + x_{1,k} \\ y_k &= H(x_k) = x_{1,k} \end{aligned}$$

where $x_k := (x_{1,k}, \ldots, x_{n,k})$, $B \in \mathbb{R}\backslash\{0\}$, and $(\partial^2\alpha/\partial x_{r+1,k}^2) \neq 0$. It can be verified by direct calculation that the IOLH $\ell_{H,F}$ is equal to $2n$ for all $1 \leq r \leq n-1$.

Example 2.13 illustrates Remark 2.7.

Example 2.13: Consider the following non-linear system:

$$\begin{aligned} x_{1,k+1} &= Bu_k \\ x_{2,k+1} &= x_{1,k} \\ &\vdots \\ x_{n-1,k+1} &= x_{n-2,k} \\ x_{n,k+1} &= \alpha(x_{p,k})x_{i,k} + \beta(x_{q,k}) \\ y_k &= H(x_k) = x_{n,k} \end{aligned}$$

where $x_k := (x_{1,k}, \ldots, x_{n,k})$, $B \in \mathbb{R}\backslash\{0\}$, $i, p, q \in \{1, \ldots, n-1\}$, $(\partial^2\alpha/\partial x_{p,k}^2) \neq 0$, and $(\partial^2\beta/\partial x_{q,k}^2) \neq 0$. The IOLH can be directly computed, obtaining $\ell_{H,F} = \min(p, q) + 1$. Note that, for $i < p$, if the control sequence $(u_0, 0, 0, \ldots)$ were used instead of the more general control sequence $(u_0, u_1, u_2, \ldots)$ in Definition 2.6, the IOLH would take the value $q + 1$, possibly different from $\ell_{H,F} = \min(p, q) + 1$.

3. Formulation of NMPC as a QP

Non-linear Model Predictive Control (NMPC) refers to a feedback control technique that, at each time step k, finds a finite sequence of control inputs $\{\bar{u}_0^*(x_k), \ldots, \bar{u}_{N-1}^*(x_k)\}$ that solves an 'underlying' fixed-horizon optimal control problem for the non-linear system (1)–(2) initialised at the current state x_k, and sets the current control input to be $u_k = \bar{u}_0^*(x_k)$, that is, equal to the first element of the optimising sequence (e.g. Mayne, Rawlings, Rao and Scokaert 2000). Different choices for the underlying optimal control problem yield controllers that can achieve different performance specifications. Here, we will consider the minimisation of a quadratic tracking cost function for system (1)–(2) under input and output constraints:

$$\min_{\{\bar{u}_0, \ldots, \bar{u}_{N-1}\}} \sum_{i=r}^{N} Q(\bar{y}_i - \bar{y}_{i,k}^d)^2 + \sum_{i=0}^{M} R\,(\bar{u}_i - \bar{u}_{i,k}^d)^2, \tag{11}$$

subject to:

$$\bar{x}_0 = x_k, \tag{12}$$

$$\bar{x}_{i+1} = F(\bar{x}_i, \bar{u}_i) \quad \text{for } i = 0, \ldots, N-1, \tag{13}$$

$$\bar{y}_i = H(\bar{x}_i) \quad \text{for } i = 0, \ldots, N, \tag{14}$$

$$W_1\bar{u}_i \leq L_1 \quad \text{for } i = 0, \ldots, M, \tag{15}$$

$$W_2\bar{y}_i \leq L_2 \quad \text{for } i = r, \ldots, N. \tag{16}$$

In (11)–(16), r is the relative degree of system (1)–(2), N is the output prediction horizon, M is the input horizon, Q and R are positive real numbers, $\{\bar{y}^d_{r,k}, \dots, \bar{y}^d_{N,k}\}$ and $\{\bar{u}^d_{0,k}, \dots, \bar{u}^d_{M,k}\}$ are desired reference sequences at time k for the system output and input to track, x_k is the current state of system (1)–(2) and W_1, L_1, W_2, L_2 are real matrices that define linear constraints on the system inputs and outputs.

Notice that, due to the non-linear dynamic and output equations (13) and (14), the cost function in (11) is not, in general, a quadratic function of the decision variables $\{\bar{u}_0, \dots, \bar{u}_{N-1}\}$. Similarly, the output constraints (16) are not, in general, linear in the decision variables. Thus, in general, it is not possible to pose the non-linear optimisation problem (11)–(16) as a QP. The following theorem gives conditions under which the latter is possible.

Theorem 3.1: *The non-linear optimisation problem* (11)–(16) *can be posed as a QP if* $\ell_{H,F} > r$ *and the output prediction horizon satisfies* $N \le \ell_{H,F} - 1$, *where* r *is the relative degree and* $\ell_{H,F}$ *is the IOLH of the output function* $H(x)$ *for system* (1)–(2).

Proof: By definition of IOLH (see Definition 2.6), $y_{\ell_{H,F}}$ is the first output sample of system (1)–(2) that is a non-linear function of the input. Thus, $y_r, \dots, y_{\ell_{H,F}-1}$ are affine functions of the decision variables $\{\bar{u}_0, \dots, \bar{u}_{N-1}\}$ and, hence, for $N \le \ell_{H,F}-1$ the cost function in (11) is quadratic in the decision variables and the output constraints (16) are linear in these variables. The result then follows. □

Note that Theorem 3.1 is applicable to general non-linear systems with arbitrary relative degree provided $\ell_{H,F} > r$ (Examples 2.11–2.13).

Our next result considers the specific case when system (1) and (2) has relative degree $r = n$. In this case, a sufficient condition can be found in Monaco and Normand-Cyrot (1987) such that there exist an invertible coordinate transformation $z_k = f(x_k)$ and a feedback law $u_k = \gamma(x_k, v_k)$ under which system (1) and (2) is transformed into the linear system

$$z_{k+1} = Az_k + Bv_k, \tag{17}$$

$$y_k = \begin{bmatrix} 1 & 0 & \dots 0 \end{bmatrix} z_k, \tag{18}$$

where the matrices A and B correspond to the Brunovsky canonical form:

$$A = \begin{bmatrix} 0 & 1 & 0 & \dots 0 \\ 0 & 0 & 1 & \dots 0 \\ & & & \vdots \\ 0 & 0 & 0 & \dots 1 \\ 0 & 0 & 0 & \dots 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}.$$

Note from Propositions 2.5 and 2.8 that the relative degree does not change after these transformations, but the IOLH is taken to ∞. Under these transformations, problem (11)–(16) takes the following form:

$$\min_{\{\bar{v}_0, \dots, \bar{v}_{N-1}\}} \sum_{i=n}^{N} Q\left(\bar{y}_i - \bar{y}^{\mathrm{d}}_{i,k}\right)^2 + \sum_{i=0}^{M} R\left(\gamma(f^{-1}(\bar{z}_i), \bar{v}_i) - \bar{u}^{\mathrm{d}}_{i,k}\right)^2, \tag{19}$$

subject to

$$\bar{z}_0 = f(x_k), \tag{20}$$

$$\bar{z}_{i+1} = A\bar{z}_i + B\bar{v}_i \quad \text{for } i = 0, \dots, N-1, \tag{21}$$

$$\bar{y}_i = \begin{bmatrix} 1 & 0 & \dots & 0 \end{bmatrix} \bar{z}_i \quad \text{for } i = 0, \dots, N, \tag{22}$$

$$W_1 \gamma(f^{-1}(\bar{z}_i), \bar{v}_i) \le L_1 \quad \text{for } i = 0, \dots, M, \tag{23}$$

$$W_2 \bar{y}_i \le L_2 \quad \text{for } i = n, \dots, N. \tag{24}$$

The following theorem gives conditions under which the non-linear optimisation problem (19)–(24) can be posed as a QP. We use in the theorem the linear horizon $\ell_{\gamma,\bar{F}}$, with $\bar{F}$ as defined in (3), which represents the number of samples it takes a new (auxiliary) input v to appear non-linearly in the original input u when a feedback law $u_k = \gamma(x_k, v_k)$ is used in system (1).

Theorem 3.2: *The non-linear optimisation problem* (19–24) *can be posed as a QP for arbitrary N if the input horizon satisfies* $M \le \ell_{\gamma,\bar{F}} - 1$, *where* $\ell_{\gamma,\bar{F}}$ *is the IOLH of the feedback law function* $\gamma(x, v)$ *for the system*

$$x_{k+1} = \bar{F}(x_k, v_k) = F(x_k, \gamma(x_k, v_k)). \tag{25}$$

Proof: By definition of IOLH (see Definition 2.6) and its invariance under coordinate transformation (Proposition 2.8), if $M \le \ell_{\gamma,\bar{F}} - 1$, then $\gamma(f^{-1}(\bar{z}_i), \bar{v}_i)$, for $i = 0, \dots, M$, are affine functions of the decision variables $\{\bar{v}_0, \dots, \bar{v}_M\}$. Thus, the cost function in (19) is a quadratic function of the decision variables and the input constraints (23) are linear in the decision variables. The result then follows. □

Remark 3.3: The results of Theorems 3.1 and 3.2 are also valid for MPC formulations that consider a quadratic term in the incremental input variables $\{(\bar{u}_0 - \bar{u}_{-1}), \dots, (\bar{u}_M - \bar{u}_{M-1})\}$. In fact, the decision variables, namely $\{\bar{u}_0, \dots, \bar{u}_M\}$, remain the same since the additional variable $\bar{u}_{-1}$ is the previous control input and it is fixed. Hence, all the affine and linear relationships between variables in both theorems, that result from considering the corresponding IOLHs, still hold.

Remark 3.4: The optimisation problems considered in Theorems 3.1 and 3.2 have only input and output constraints. State constraints can be handled in a similar way by using the concept of ISLH introduced in Definition 2.9.

Remark 3.5: The results of Theorems 3.1 and 3.2 impose limits on the values of N (in the case of Theorem 3.1) or M (in the case of Theorem 3.2). It is well known that this *shortsightedness* in MPC can compromise stability of the scheme. Robust stability of the resulting NMPC scheme with respect to unknown disturbances can be addressed by incorporating additional *stability constraints*. In the context of Theorem 3.1, and for regulation problems, a possible choice of additional stability constraints to be appended to (12)–(16) is

$$\|P_V F(\bar{x}_0, \bar{u}_0)\|_\infty - \|P_V \bar{x}_0\|_\infty \leq -\|Q_V \bar{x}_0\|_\infty \tag{26}$$

where P_V and Q_V are full-column rank matrices that can be determined; see details in Mare, Lazar and De Doná (2007) where the concept of ISLH in Definition 2.9 is used. In the latter reference, it is shown that the inequality constraints (26) can be expressed as *linear* inequalities in the variable $\bar{u}_0$. Thus, the simplicity of the solution afforded by the QP formulation is retained and, in addition, the scheme possesses an *a priori* input-to-state stability guarantee.

Remark 3.6: The class of problems that Theorem 3.2 deals with (namely, feedback linearisable ones) is particularly well suited to be treated with the related concept of *flatness*, which provides a useful methodology for trajectory tracking problems. For example, flatness is used in Sira-Ramírez and Castro-Linares (2000) to solve trajectory tracking problems for a discretised non-linear flexible joint robot. Such example will be the subject of the case study in the following section.

4. Case study

In this section, we consider an example of a non-linear system for which NMPC can be solved via QP using the concept of IOLH and the results of the previous sections.

4.1. The flexible joint robot

Consider the following non-linear state-space model of a flexible joint robot, which has been taken from Sira–Ramírez and Castro–Linares (2000):

$$x_{1,k+1} = x_{1,k} + T_s x_{2,k}, \tag{27a}$$

$$x_{2,k+1} = x_{2,k} + \frac{MgLT_s}{I}\sin(x_{1,k}) + \frac{K_a T_s}{I}(x_{1,k} - x_{3,k}), \tag{27b}$$

$$x_{3,k+1} = x_{3,k} + T_s x_{4,k}, \tag{27c}$$

$$x_{4,k+1} = x_{4,k} + \frac{K_a T_s}{J}(x_{1,k} - x_{3,k}) + \frac{T_s}{J}u_k, \tag{27d}$$

$$y_k = x_{1,k} \tag{28}$$

where $x_{1,k}$ is the link angular position, $x_{2,k}$ is the link angular velocity, $x_{3,k}$ is the motor axis angular position, $x_{4,k}$ is the motor axis angular velocity, and u_k is the motor applied torque. The parameters I, J, MgL and K_a represent the inertia of the link, the motor inertia, the nominal load in the link, and the flexible joint stiffness coefficient, respectively, and T_s is the sampling period. Functions F and H can be readily defined such that system (27) and (28) takes the form (1) and (2), with $\mathbb{M} = \mathbb{R}^4$.

Our goal is to design a controller such that the angular position of the link, namely $y_k = H(x_k) = x_{1,k}$, tracks a given arbitrary reference signal $\{y_k^d\}$. In addition, the applied torque u_k is required to remain in the constraint set $\Omega = [-2, 2]$.

Using the definitions in §2 it can be seen that system (27) has relative degree $r = n = 4$, and that the IOLH $\ell_{H,F}$ is equal to 6. The model for the flexible joint manipulator (27) can be expressed in normal form (Monaco and Normand–Cyrot 1987) through a coordinate transformation $z_k = f(x_k)$, with

$$\begin{aligned}
z_{1,k} &= x_{1,k},\\
z_{2,k} &= x_{1,k} + T_s x_{2,k},\\
z_{3,k} &= x_{1,k} + 2T_s x_{2,k} + \frac{MgLT_s^2}{I}\sin(x_{1,k})\\
&\quad + \frac{K_a T_s^2}{I}(x_{1,k} - x_{3,k}),\\
z_{4,k} &= x_{1,k} + 3T_s x_{2,k} + 2\frac{MgLT_s^2}{I}\sin(x_{1,k})\\
&\quad + 2\frac{K_a T_s^2}{I}(x_{1,k} - x_{3,k}) + \frac{MgLT_s^2}{I}\sin(x_{1,k} + T_s x_{2,k})\\
&\quad + \frac{K_a T_s^2}{I}(x_{1,k} + T_s x_{2,k} - x_{3,k} - T_s x_{4,k}).
\end{aligned}$$

Straightforward calculations based on simple substitution yield

$$z_{1,k+1} = z_{2,k}, \tag{29a}$$

$$z_{2,k+1} = z_{3,k}, \tag{29b}$$

$$z_{3,k+1} = z_{4,k}, \tag{29c}$$

$$
\begin{aligned}
z_{4,k+1} = \frac{K_aT_s^4}{JI}\bigg(u_k + \left[MgL + \frac{JMgL}{K_aT_s^2}\right]\sin(z_{1,k}) \\
- \frac{JMgL}{K_aT_s^2}[2\sin(z_{2,k}) - \sin(z_{3,k})]\bigg) \\
- \frac{K_aT_s^4}{JI}\frac{J+I}{T_s^2}[z_{1,k} - 2z_{2,k} + z_{3,k}] \\
- [z_{1,k} - 4z_{2,k} + 6z_{3,k} - 4z_{4,k}]. \quad (29d)
\end{aligned}
$$

It can be seen from (29) that the flexible joint robot model is state feedback linearisable. The feedback law $u_k = \gamma(x_k, v_k) = \gamma(f^{-1}(z_k), v_k)$, with

$$
\begin{aligned}
\gamma(f^{-1}(z_k), v_k) := \frac{JI}{K_aT_s^4}v_k + \frac{1}{K_aT_s^4} \\
\times \begin{bmatrix} (J+I)(K_aT_s^2) + JI \\ -2(J+I)(K_aT_s^2) - 4JI \\ (J+I)(K_aT_s^2) + 6JI \\ -4JI \end{bmatrix}^T \begin{bmatrix} z_{1,k} \\ z_{2,k} \\ z_{3,k} \\ z_{4,k} \end{bmatrix} \\
+ \frac{MgL}{K_aT_s^2}\begin{bmatrix} -(K_aT_s^2 + J) \\ 2J \\ -J \end{bmatrix}^T \begin{bmatrix} \sin(z_{1,k}) \\ \sin(z_{2,k}) \\ \sin(z_{3,k}) \end{bmatrix}, \quad (30)
\end{aligned}
$$

transforms the system into the *linear* system described in (17) and (18). We denote[3] the linearised system represented by Equation (17) as $z_{k+1} = \bar{F}(z_k, v_k)$. The IOLH of the function γ under the linearised dynamics is $\ell_{\gamma,\bar{F}} = 2$.

We aim at obtaining the controller as the solution of the optimisation problem (19)–(23), where $W_1 = [1\ -1]^T$ and $L_1 = [2\ 2]^T$. We use $N = 8$ and $M = 1$ to solve the optimisation problem, the former is chosen arbitrarily while the latter is given by $M = \ell_{\gamma,\bar{F}} - 1$. Theorem 3.2 then guarantees that the problem can be posed as a QP. To illustrate this further, note that the feedback transformation (30) maps the linear inequality constraint $u_k \in [-2, 2]$ onto a linear inequality constraint on v_k for $k < \ell_{\gamma,\bar{F}} = 2$, since u_0 and u_1 are affine functions of v_0 and v_1:

$$
\begin{aligned}
u_0 &= \gamma(f^{-1}(z_0), v_0) = K_0v_0 + \psi_0(z_0), \\
u_1 &= \gamma(f^{-1}(z_1), v_1) = K_1\begin{bmatrix} v_0 \\ v_1 \end{bmatrix} + \psi_1(z_0),
\end{aligned}
$$

where the constants K_0, K_1, and the non-linear functions ψ_0, ψ_1 can be readily obtained from (30) and (17). In addition, note that for $k = \ell_{\gamma,\bar{F}} = 2$ the transformation between u_k and v_k given by (30) is no longer affine, since

$$
u_2 = \gamma(f^{-1}(z_2), v_2) = K_2\begin{bmatrix} v_0 \\ v_1 \\ v_2 \end{bmatrix} + \psi_2(z_0) + K_3\sin(v_0),
$$

with constants K_2, K_3 and non-linear function ψ_2 obtained from (30) and (17), $K_3 \neq 0$. (Notice that it is in the term $\sin(z_{3,k})$ in (30) where the affine relationship between u_k and v_k is broken for the first time since $z_{3,2} = v_0$.) Hence, $u_2 \in [-2, 2]$ is not transformed into a linear inequality constraint on v_k.

Given the desired output reference $\bar{y}_{i,k}^d$, the input reference $\bar{u}_{i,k}^d$ can be constructed as follows: an input reference $\bar{v}_{i,k}^d$ for the linearised system is constructed first, which from the Brunovsky structure (17) and (18) of the linearised system is simply $\bar{v}_{i,k}^d = \bar{y}_{i+4,k}^d$ (i.e. the output reference four samples ahead). Then, using (30) it is straightforward to obtain the input reference $\bar{u}_{i,k}^d$ which depends on $\bar{v}_{i,k}^d$ and the (known) current state $z_k = f(x_k)$.

4.2 *Simulations*

Figure 1 shows the time evolution of the input torque, u_k, and the input of the linearised system, v_k, resulting from the receding horizon implementation of the solution obtained by solving a QP, as described above. The weights on the tracking error and on the deviation of the input from the input reference are chosen as $Q = 1$, $R = 3.5$, respectively, the sampling time is set to $T_s = 0.1$ s, and the system is required to track a square wave of amplitude $\pi/2$ and period 4 s. The evolution of the state and the output are shown in Figure 2. It can be seen from both figures that the solution satisfies the constraints, while the output tracks the given reference signal. For a comparison with standard methods for NMPC using non-linear programming, see Mare and De Doná (2006) where a simulation experiment is performed and the on-line computational times required by an explicit QP solution are compared to those required by a numerical optimisation routine. The computational times of the explicit QP solution were in all cases much smaller than those of the numerical optimisation routine.

5. Conclusions

This article has investigated conditions under which the optimal control sequence for a constrained non-linear system can be computed via Quadratic Programming (QP). The main contribution has been to show that the optimal control sequence for non-linear single-input single-output systems, with a quadratic cost and linear inequality constraints can be computed in exact form via QP provided the optimisation horizon is no larger than a critical quantity that we name the IOLH. We have also shown that the IOLH is greater than or equal to the

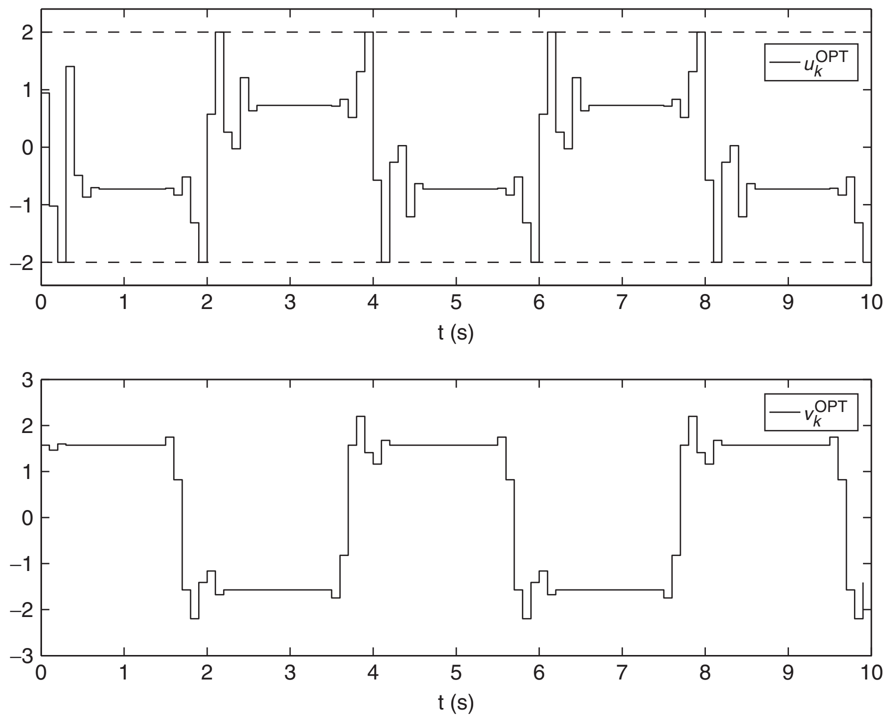

Figure 1. Receding horizon implementation. Upper plot: control input u_k^{opt} and limits (dashed line) of the constraint set. Lower plot: auxiliary control input v_k^{opt} for the linearised dynamics.

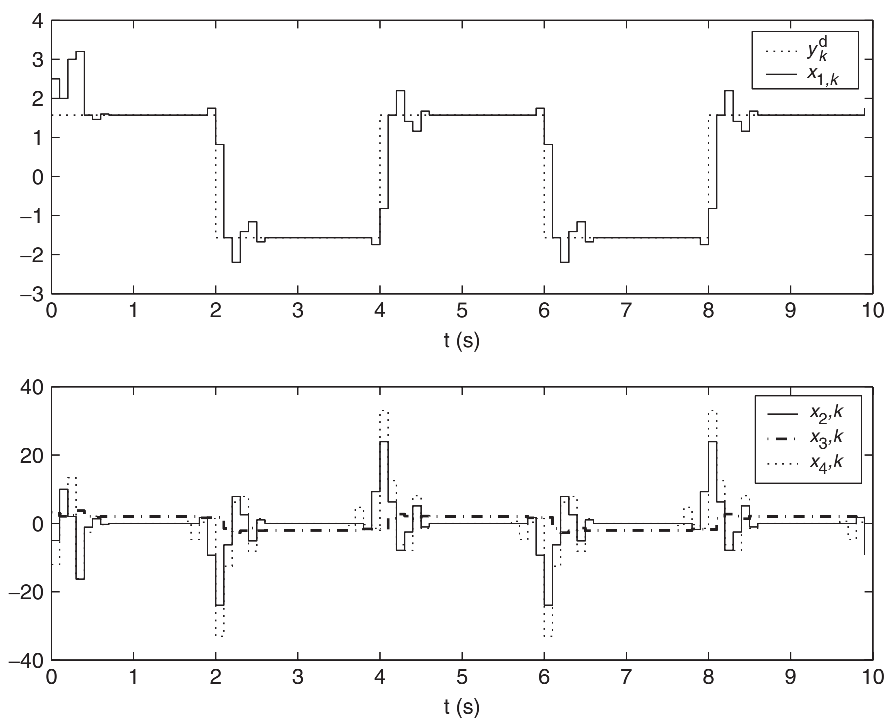

Figure 2. Receding horizon implementation. Evolution of the original state x_k and the output $y_k = x_{1,k}$ (y_k^d is the output reference).

system relative degree and is invariant under coordinate transformation but in general not under feedback. The results provided can contribute to more efficient ways of applying MPC to non-linear systems.

Notes

1. Note that $\ell_{H,F}$ is used to simplify the notation. Strictly, $\ell_{\tilde{H},F}$ should be used, where $\tilde{H}(\cdot, u) := H(\cdot)\ \forall u \in \mathbb{R}$.
2. Note that since feedback is used, the system is now $\bar{F}$. The output function H, however, remains the same.
3. Note that, strictly, we should have used the notation $z_{k+1} = \tilde{\bar{F}}(z_k, v_k)$ since both, a feedback transformation and a coordinate transformation have been performed (cf (3) and (4)). However, the adopted notation simplifies the exposition.

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