

## A COMPACTLY GENERATED GROUP WHOSE HECKE ALGEBRAS ADMIT NO BOUNDS ON THEIR REPRESENTATIONS

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**Abstract.** We construct a compactly generated, totally disconnected, locally compact group whose Hecke algebra with respect to any compact open subgroup does not have a  $C^*$ -enveloping algebra.

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**1. Introduction.** A group-subgroup pair  $(G, C)$  is called a *Hecke pair* if  $C$  is almost normal in  $G$ , i.e. if every double coset of  $C$  in  $G$  is a union of a finite number of left cosets of  $C$  in  $G$ . To a Hecke pair one associates a convolution algebra of functions,  $\mathbb{C}[G, C]$ , on the space of double cosets of  $C$  in  $G$  which coincides with the group algebra  $\mathbb{C}[G/C]$  when  $C$  is normal, see §2 and [6]. The algebra  $\mathbb{C}[G, C]$  is called the *Hecke algebra* of the pair  $(G, C)$ . Just as representations of a group  $G$  on a complex vector space correspond to representations of its group algebra  $\mathbb{C}[G]$ , the algebra representations of  $\mathbb{C}[G, C]$  correspond to linear representations of  $G$  which are generated by their  $C$ -fixed vectors [6, § 3.1] or [8, Chapter 1].

There is a  $*$ -algebraic structure on  $\mathbb{C}[G, C]$ , which is induced by inversion of group elements in  $G$ . This paper addresses the question of how well unitary representations of a group  $G$  which are topologically generated by  $C$ -fixed vectors correspond to  $*$ -representations of  $\mathbb{C}[G, C]$ . It is known that there is not always an exact correspondence. This is in contrast with the situation for the group algebra  $\mathbb{C}[G]$ , which corresponds to the special case  $C = \{1\}$ . For an arbitrary almost normal subgroup  $C$ , the Hecke algebra  $\mathbb{C}[G, C]$  may have many  $*$ -representations that are not induced by unitary representations of  $G$ . This will be the case if  $\mathbb{C}[G, C]$  does not have an enveloping  $C^*$ -algebra because the  $*$ -representations arising from unitary representations of  $G$  are uniformly bounded on each element of  $\mathbb{C}[G, C]$ .

If  $G$  is the infinite dihedral group and  $C$  is a subgroup of order 2 then  $\mathbb{C}[G, C]$  does not have an enveloping  $C^*$ -algebra, see [10, Example 3.1] or Lemma 3.2. A less elementary example, where  $G$  is  $SL_2(\mathbb{Q}_p)$  and  $C$  is a Borel subgroup, is discussed in [6]. In each of these negative examples, replacing  $C$  by a finite index subgroup (the trivial group in the first case and an Iwahori subgroup in the second, see [6]) produces a Hecke algebra which does have an enveloping  $C^*$ -algebra.

If some almost normal subgroup  $C$  in  $G$  is given, can we always choose an almost normal subgroup  $C'$  within the same commensurability class such that  $\mathbb{C}[G, C']$  has an enveloping  $C^*$ -algebra? We are interested in the case where  $G$  is totally disconnected,

locally compact group and  $C$  is a compact, open subgroup. Such a group-subgroup pair  $(G, C)$  is a Hecke pair because every double coset of  $C$  in  $G$  is a compact set with an open cover of left cosets. If  $C$  is topologically finitely generated then the commensurability class of  $C$  consists of all compact open subgroups of  $G$  by the main result of [9], Théorème 0.1. Moreover all Hecke algebras can be realised by such a topological Hecke pair in the following sense. Given a Hecke pair  $(G', C')$  we can find a Hecke pair  $(G, C)$ , called its *Schlichting completion*, consisting of a totally disconnected, locally compact topological group and a compact open subgroup with the property that  $\mathbb{C}[G, C] \cong \mathbb{C}[G', C']$  as  $*$ -algebras, see [5, 7] and [10, section 4].

Our question then becomes: Which conditions on a totally disconnected, locally compact group  $G$  ensure that there is some compact open subgroup  $C$  in  $G$  such that  $\mathbb{C}[G, C]$  has an enveloping  $C^*$ -algebra? In Lemma 2.5 below we establish that for a fixed element  $x \in G$  one can find a compact open subgroup  $C$  of  $G$  with the property that the double coset  $Cx C$  has a finite norm in every  $*$ -representation of  $\mathbb{C}[G, C]$  by choosing  $C$  to be sufficiently small. This suggests that  $\mathbb{C}[G, C]$  may have an enveloping  $C^*$ -algebra whenever the compact open subgroup  $C$  is small enough provided the result of Lemma 2.5 can be obtained uniformly for all  $x \in G$ .

In § 3 we construct a compactly generated group for which the results of Lemma 2.5 can not be obtained uniformly. More precisely, our example shows that compact generation does not ensure the existence of a compact open subgroup such that the corresponding Hecke algebra has an enveloping  $C^*$ -algebra.

In the final section we present topological conditions on  $G$  which exclude any example with similar features to the one constructed here. These conditions translate into conditions on an algebraic Hecke pair via the Schlichting completion. We conjecture that these conditions ensure existence of a largest  $C^*$ -norm on Hecke algebras with respect to sufficiently small compact open subgroups.

**2. Hecke Algebras.** The Hecke algebra of a Hecke pair  $(G, C)$  can be defined by a product formula that involves counting the number of left  $C$ -cosets in each double coset  $Cx C$ . If  $G$  is a totally disconnected, locally compact topological group and  $C$  is a compact open subgroup, then every double coset of  $C$  in  $G$  is a compact set with an open cover of left cosets. Hence every double coset of  $C$  in  $G$  is a union of a finite number of left cosets of  $C$  in  $G$ .

When  $G$  is locally compact and  $C$  is a compact open subgroup, the Hecke algebra  $\mathbb{C}[G, C]$  can be realised as a subalgebra of the measure algebra on  $G$  with the familiar convolution product. We shall adopt this point of view because the formulas are simpler with this approach.

If  $G$  is a topological group, denote by  $\mathcal{M}G$  the convolution algebra of compactly supported bounded complex measures on  $G$ . If  $C$  is a compact group, denote by  $m_C$  its normalized Haar measure. Moreover, we denote both an element of  $G$  and its point mass in  $\mathcal{M}G$  by the same symbol.

**DEFINITION 2.1.** Suppose  $G$  is a totally disconnected, locally compact topological group and let  $C$  be a compact open subgroup. The **Hecke algebra** of  $(G, C)$ ,  $\mathbb{C}[G, C]$ , is the subalgebra of  $\mathcal{M}G$  consisting of the measures with constant density on double cosets modulo  $C$ , i.e.  $\mathbb{C}[G, C] = m_C * \mathcal{M}G * m_C$ . The map sending a measure  $\mu$  to the complex conjugate of the image of  $\mu$  under inversion defines an involution  $*$  on  $\mathcal{M}G$  which restricts to  $\mathbb{C}[G, C]$ , because  $m_C^* = m_C$ .

The element  $m_C$  is an identity for the algebra  $\mathbb{C}[G, C]$ .

When proving our main result, we reduce a statement about the Hecke pair  $(G, C)$  to a statement about another Hecke pair of the form  $(H, C')$  with  $H \leq G$  and  $C' \geq C$ . For this task, the following observations will be useful.

LEMMA 2.2. *Suppose  $G$  is a topological group,  $C$  is a compact open subgroup of  $G$ , and  $H$  is a subgroup of  $G$  containing  $C$ . The inclusion of  $H$  in  $G$  induces an injective unital  $*$ -homomorphism  $\mathbb{C}[H, C] \rightarrow \mathbb{C}[G, C]$ . We may therefore view  $\mathbb{C}[H, C]$  as a unital  $*$ -subalgebra of  $\mathbb{C}[G, C]$ .  $\square$*

LEMMA 2.3. *Suppose  $G$  is a topological group,  $C$  is a compact open subgroup of  $G$ , and  $C'$  is a compact subgroup of  $G$  containing  $C$ . Then  $\mathbb{C}[G, C']$  is a  $*$ -subalgebra of  $\mathbb{C}[G, C]$ .*

*Proof.* We have  $m_{C'} * m_C = m_{C'} = m_C * m_{C'}$  and hence

$$\begin{aligned} \mathbb{C}[G, C'] &= m_{C'} * \mathcal{M}G * m_{C'} \\ &= m_{C'} * m_C * \mathcal{M}G * m_C * m_{C'} \\ &= m_{C'} * \mathbb{C}[G, C] * m_{C'}. \end{aligned}$$

Since  $m_{C'}^* = m_{C'}$ ,  $\mathbb{C}[G, C']$  is  $*$ -closed in  $\mathbb{C}[G, C]$ .  $\square$

To concisely formulate our results we introduce the following notation.

NOTATION 2.4. *For an element  $m \in \mathbb{C}[G, C]$  we let*

$$\|m\|_{G,C} := \sup\{\|\pi(m)\| : \pi \text{ is a } * \text{-representation of } \mathbb{C}[G, C] \text{ on a Hilbert space}\}.$$

There is an enveloping  $C^*$ -algebra for  $\mathbb{C}[G, C]$  iff  $\|m\|_{G,C} < \infty$  for all  $m$  in  $\mathbb{C}[G, C]$ .

The following result shows that a technique of descent enables us to bound norms on any chosen double coset. It motivates the suspicion, mentioned in the introduction, that for well-behaved groups the Hecke algebra with respect to a sufficiently small compact open subgroup has an enveloping  $C^*$ -algebra.

LEMMA 2.5. *Let  $G$  be a topological group and let  $U$  and  $V$  be compact open subgroups of  $G$ . Then  $\|\sum_{i=1}^n \lambda_i m_V * x_i * m_V\|_{G,V} < \infty$  provided that  $V$  satisfies*

$$V \cup \bigcup_{i=1}^n x_i^{-1} V x_i \subseteq U.$$

*Proof.* It suffices to show that  $\|m^* * m\|_{G,V} < \infty$  for all elements of the form  $m = m_V * x_i * m_V \in \mathbb{C}[G, V]$ .

By assumption  $V x_i^{-1} V x_i V \subseteq U$ , showing that  $m^* * m$  belongs to the algebra  $\mathbb{C}[U, V]$ . Since  $U$  is compact,  $\mathbb{C}[U, V]$  is finite dimensional. Thus  $\|m^* * m\|_{U,V} < \infty$  and hence  $\|m^* * m\|_{G,V} < \infty$  as well.

As a consequence of Lemma 2.5 we obtain the following result.

PROPOSITION 2.6. *Suppose that  $G$  is topological group. Let  $U$  be a compact, open, normal subgroup of  $G$ . Then for any compact, open subgroup  $V \subseteq U$  the algebra  $\mathbb{C}[G, V]$  has an enveloping  $C^*$ -algebra.*

*Proof.* Lemma 2.5 shows that for any  $x$  in  $G$   $\|m_V * x * m_V\|_{G,V} < \infty$  as soon as  $V \cup x^{-1} V x \subseteq U$ , a condition which is automatically satisfied by the normality

assumption on  $U$ . The Hecke algebra  $\mathbb{C}[G, V]$  is spanned by elements of the form  $m_V * x * m_V$  and thus the conclusion follows.  $\square$

The question arises as to whether we can relax the condition on  $U$  somewhat by making it dependent on the element  $x$  under consideration. More precisely, we pose the following question.

**OPEN QUESTION 2.7.** *Suppose that  $G$  is a compactly generated, totally disconnected, locally compact group which is uniscalar in the sense that for each  $x \in G$  there is a compact, open subgroup  $U$  such that  $x^{-1}Ux = U$ . Does  $\mathbb{C}[G, V]$  have an enveloping  $C^*$ -algebra if  $V$  is sufficiently small?*

**3. The Construction.** In this section we provide an example showing that a compactly generated group  $G$  need not admit a largest  $C^*$ -norm on the Hecke algebra with respect to any compact open subgroup.

Our example is built from a Hecke pair  $(D, F)$  with  $F$  finite such that  $\mathbb{C}[D, F]$  contains an element  $\Lambda_1$  such that  $\|\Lambda_1\|_{D,F} = \infty$ . The group  $G$  will be assembled from  $D$  in such a way that infinitely many copies of  $\mathbb{C}[D, F]$  survive in  $\mathbb{C}[G, C]$  for any compact open subgroup  $C$  of  $G$ .

Let  $D := \mathbb{Z} \rtimes F$  be the semidirect product where  $F$  is the group of order 2 acting on  $\mathbb{Z}$  by inversion, so that  $D$  is isomorphic to the infinite dihedral group. Equip  $D$  with the discrete topology. Next let  $H$  be the restricted product

$$H := \prod_{\mathbb{Z}} D|F := \{h \in D^{\mathbb{Z}} \mid h(n) \in F \text{ for all but finitely many } n\}.$$

Recall that the restricted product  $\prod_{i \in I} G_i|O_i$ , where  $G_i$  are locally compact groups and  $O_i$  is a compact, open subgroup of  $G_i$  has a topology defined by a basis of the identity consisting of  $\prod_{i \in i} U_i$  with  $U_i$  open in  $G_i$  for all  $i$  and  $U_i = O_i$  for all but finitely many  $i$ . The group  $H$  meets all our requirements except that of compact generation.

There is an action of  $\mathbb{Z}$  on  $H$  via the shifting of indices. The semidirect product  $G = \mathbb{Z} \rtimes H$  with respect to this action is our counterexample.

The rest of this section is devoted to the proof of the following result.

**THEOREM 3.1.** *The group  $G$  is compactly generated and, for any compact open subgroup  $C$  in  $G$ , there is an element  $\Lambda_C$  of  $\mathbb{C}[G, C]$  such that  $\|\Lambda_C\|_{G,C} = \infty$ .*

The existence of  $\Lambda_C$  will be shown by reduction to the corresponding statement for  $\mathbb{C}[H, C]$ , which in turn will be reduced to part (4) of Lemma 3.2 by Lemma 3.4.

We begin by establishing some notation. The elements of  $F$  will be denoted by  $\bar{0}$  and  $\bar{1}$ , with the group operation being addition modulo 2, and we identify  $F$  with the subgroup  $\{(0, \bar{a}) \mid \bar{a} \in F\}$  of  $D$ . The group  $K := \prod_{\mathbb{Z}} F$  is naturally identified with a compact open subgroup of  $H$ , and hence of  $G$ . Since  $F$  is a maximal compact subgroup of  $D$ ,  $K$  is a maximal compact subgroup of  $G$ . The identity element of  $H$  will be denoted by  $\mathbf{0}$  and the shift of  $H$  by 1 will be denoted by  $\sigma$ . The elements of  $G$  will be written as  $\sigma^l h$  with  $l \in \mathbb{Z}$  and  $h \in H$ . We will define conjugation such that, for  $l \in \mathbb{Z}$ , we have  $\sigma^l(h) = \sigma^l h \sigma^{-l}$ .

That  $G$  is compactly generated may be seen by verifying that it is generated by  $K \cup \{\sigma, h_0\}$ , where  $h_0$  is the function in  $H$  which takes the value  $(1, \bar{0})$  at 0 and the identity elsewhere. It remains to prove the assertion about unboundedness of representations;

our proof relies on the corresponding statement about the Hecke algebra  $\mathbb{C}[D, F]$ , which we establish first.

The double cosets over  $F$  in  $D$  are  $F(n, \bar{0})F = \{(n, \bar{0}), (-n, \bar{0}), (n, \bar{1}), (-n, \bar{1})\}$ , where  $n \in \mathbb{N}$ . This and elementary calculations then yield the following description of the structure of  $\mathbb{C}[D, F]$ . The Chebychev polynomials are defined in [4], page 123.

LEMMA 3.2. *Let  $\Lambda_n = m_F * (n, \bar{0}) * m_F$ ; then*

1.  $\mathbb{C}[D, F]$  is the linear span of  $\{\Lambda_n \mid n \in \mathbb{N}\}$  and for all natural numbers  $m$  and  $n$  we have

$$\begin{aligned} \Lambda_n^* &= \Lambda_n \\ \Lambda_m * \Lambda_n &= \frac{1}{2} (\Lambda_{m+n} + \Lambda_{|m-n|}). \end{aligned}$$

2.  $\mathbb{C}[D, F]$  is isomorphic to the ring of polynomials in  $\Lambda_1$ , and  $\Lambda_n = p_n(\Lambda_1)$ , where  $p_n$  is the degree  $n$  Chebychev polynomial.
3. For any real number  $\lambda$  the assignment  $\pi_\lambda: \Lambda_n \mapsto p_n(\lambda)$  defines a one-dimensional unital  $*$ -representation of  $\mathbb{C}[D, F]$ .
4. The supremum  $\sup\{\|\pi_\lambda(\Lambda_1)\|: \lambda \in \mathbb{R}\}$  is infinite. In particular  $\|\Lambda_1\|_{D,F} = \infty$ .

Theorem 3.1 will follow from this Lemma because the group  $G$  is assembled from  $D$  and  $F$  in such a way that infinitely many copies of  $\mathbb{C}[D, F]$  survive in  $\mathbb{C}[G, C]$  for any compact open subgroup  $C$  of  $G$ , as we will see shortly.

Consider first the structure of  $\mathbb{C}[G, C]$  when  $C = K$ . The set of probability measures  $\{m_K * \sigma^l h * m_K: \sigma^l h \in G\}$  spans  $\mathbb{C}[G, K]$  and, since  $m_K$  is invariant under the shift on  $H$ ,  $m_K * \sigma^l h * m_K = \sigma^l * m_K * h * m_K$ . Hence  $\mathbb{C}[G, K]$  has the  $\mathbb{Z}$ -grading

$$\mathbb{C}[G, K] = \bigoplus_{l \in \mathbb{Z}} \sigma^l * \mathbb{C}[H, K].$$

To decompose  $\mathbb{C}[H, K]$ , define, for each finite  $I \subset \mathbb{Z}$ ,

$$H_I = \{h \in H \mid h(n) = (0, \bar{0}) \text{ unless } n \in I\}.$$

The group  $H_{\{i\}}$  will be written as  $H_i$ . Since  $H_I$  is the direct product  $\prod_{i \in I} H_i$  and each  $H_i$  is isomorphic to  $D$ , we have

$$\mathbb{C}[H_I K, K] \cong \bigotimes_{i \in I} \mathbb{C}[H_i K, K] \text{ and } \mathbb{C}[H_i K, K] \cong \mathbb{C}[D, F] \text{ for each } i, \tag{1}$$

by [8, Theorem 6.3], where the involution on the tensor product is the tensor power of the involution on  $\mathbb{C}[D, F]$ . For index sets  $I$  and  $J$  with  $I \subseteq J$  the inclusion homomorphisms  $\mathbb{C}[H_I K, K] \rightarrow \mathbb{C}[H_J K, K]$  are unital and

$$\mathbb{C}[H, K] = \varinjlim_{I \text{ finite}} \mathbb{C}[H_I K, K]. \tag{2}$$

The combined isomorphism  $\mathbb{C}[H_I K, K] \rightarrow \bigotimes_{i \in I} \mathbb{C}[D, F]$  in (1) is given by

$$m_K * h * m_K \mapsto \bigotimes_{i \in I} m_F * h(i) * m_F \quad (h \in H_I). \tag{3}$$

This, plus the fact that conjugation by  $\sigma$  intertwines this family of isomorphisms as the shift of the index set  $I$ , allow us to define a one-dimensional  $*$ -representation of  $\mathbb{C}[G, K]$  by taking the same representation for each factor  $\mathbb{C}[D, F]$  and the trivial representation for the shift  $\sigma$ .

LEMMA 3.3. *Let  $\lambda \in \mathbb{R}$  and define*

$$\varpi_\lambda(\sigma^l * m_K * h * m_K) = \prod_{n \in \mathbb{Z}} \pi_\lambda(m_F * h(n) * m_F); \quad \sigma^l h \in G,$$

where  $\pi_\lambda$  is the  $*$ -representation of  $\mathbb{C}[D, F]$  defined in Lemma 3.2. Then  $\varpi_\lambda$  is a unital  $*$ -representation of  $\mathbb{C}[G, K]$ .

(Note the infinite product on the right hand side is well-defined because all but finitely many terms are equal to 1.)

Lemma 3.3 and 3.2 imply that when  $C = K$ , we may take

$$\Lambda_K := m_K * h_0 * m_K$$

in Theorem 3.1 (Recall that  $h_0 \in H$  is defined by  $h_0(0) = (1, \bar{0})$  and  $h_0(n) = (0, \bar{0})$  for  $n \neq 0$ .)

Next consider the case when  $C$  is a proper compact open subgroup of  $K$ . In this case  $m_C$  is not invariant under the shift on  $H$ , that is,  $m_{\sigma(C)} \neq m_C$ . Denote the product of probability measures  $m_C * m_{\sigma(C)}$  by  $m_C^{(l)}$ . Then  $m_C^{(l)}$  is the Haar measure of the subgroup  $C + \sigma^l(C)$  of  $K$ .

The next lemma shows that if  $C$  is a proper, compact, open subgroup of  $K$ , then  $\mathbb{C}[H, C]$  has quotients isomorphic to  $\mathbb{C}[D, F]$ . Define  $H_{<i}$  and  $H_{>j}$  to be the subgroups of  $H$  supported on the coordinates at most  $i$  and at least  $j$  respectively;

$$H_{<i} := \{h \in H \mid h(n) = (0, \bar{0}) \text{ if } n \geq i\} \text{ and } H_{>j} := \{h \in H \mid h(n) = (0, \bar{0}) \text{ if } n \leq j\}.$$

Define subgroups  $K_{<i}$  and  $K_{>j}$  of  $K$  similarly. Note that the Hecke algebras  $\mathbb{C}[H_{<i}, K_{<i}]$  and  $\mathbb{C}[H_{>j}, K_{>j}]$  are both isomorphic to  $\mathbb{C}[H, K]$ , which has  $\mathbb{C}[D, F]$  as a quotient; hence  $\mathbb{C}[H, C]$  has  $\mathbb{C}[D, F]$  as a quotient as well.

LEMMA 3.4. *Let  $C$  be a proper open subgroup of  $K$ . Then, for some interval  $[i, j] \subseteq \mathbb{Z}$ , there is a surjective  $*$ -homomorphism*

$$T: \mathbb{C}[H, C] \rightarrow \mathbb{C}[H_{<i}, K_{<i}] \otimes \mathbb{C}[H_{>j}, K_{>j}]$$

such that  $T(m_C^{(l)}) = 0$  for every  $l \neq 0$ .

*Proof.* Since  $C$  is open, there are integers  $i \leq j$  such that

$$\{k \in K \mid k(n) = \bar{0} \text{ for } i \leq n \leq j\}$$

is contained in  $C$ . Since  $C$  is a proper subgroup of  $K$ , we may choose  $i$  to be maximal and  $j$  to be minimal with respect to this property.

Put  $C_{[i,j]} := C \cap H_{[i,j]}$ . Then  $H = H_{<i} \times H_{[i,j]} \times H_{>j}$  and  $C = K_{<i} \times C_{[i,j]} \times K_{>j}$  and it follows that

$$\mathbb{C}[H, C] \cong \mathbb{C}[H_{<i}, K_{<i}] \otimes \mathbb{C}[H_{[i,j]}, C_{[i,j]}] \otimes \mathbb{C}[H_{>j}, K_{>j}]. \tag{4}$$

Writing  $h|_{<i}$ ,  $h|_{[i,j]}$  and  $h|_{>j}$  for the projections of  $h \in H$  onto  $H_{<i}$ ,  $H_{[i,j]}$  and  $H_{>j}$  respectively, the isomorphism in (4) is given by

$$m_C * h * m_C \mapsto (m_{K_{<i}} * h|_{<i} * m_{K_{<i}}) \otimes (m_{C_{[i,j]}} * h|_{[i,j]} * m_{C_{[i,j]}}) \otimes (m_{K_{>j}} * h|_{>j} * m_{K_{>j}}).$$

The homomorphism  $T$  we seek will be obtained by composing this isomorphism with  $\text{Id} \otimes \phi \otimes \text{Id}$  for a non-zero multiplicative linear functional  $\phi: \mathbb{C}[H_{[i,j]}, C_{[i,j]}] \rightarrow \mathbb{C}$  that remains to be constructed.

Functionals on  $\mathbb{C}[H_{[i,j]}, C_{[i,j]}]$  are determined by characters on  $H_{[i,j]}$  and these characters are in their turn indexed by functions  $k: [i, j] \rightarrow \{\pm 1\}$  as follows. First note that the character group of  $F = \{\bar{0}, \bar{1}\}$  may be identified with  $(\{\pm 1\}, \times)$  by

$$\langle \bar{a}, \pm 1 \rangle = \begin{cases} 1, & \text{if } \bar{a} = \bar{0} \\ \pm 1, & \text{if } \bar{a} = \bar{1} \end{cases},$$

and let  $q: D \rightarrow F$  be the homomorphism  $q(n, \bar{a}) = \bar{a}$ . Then for each  $k \in \{\pm 1\}^{[i,j]}$  the map  $\chi_k: H_{[i,j]} \rightarrow \{\pm 1\}$  defined by

$$\chi_k(h) = \prod_{n \in [i,j]} \langle q \circ h(n), k(n) \rangle$$

is a character. Next, define the linear functional  $\phi_k$  on  $\mathbb{C}[H_{[i,j]}, C_{[i,j]}]$  by

$$\phi_k(\mu) = \int_{H_{[i,j]}} \chi_k(h) d\mu(h) \quad \text{for } \mu \in \mathbb{C}[H_{[i,j]}, C_{[i,j]}].$$

Then  $\phi_k$  is multiplicative and is non-zero if  $C_{[i,j]} \subseteq \ker \chi_k$  because  $\phi(m_{C_{[i,j]}}) = 1$  in that case. We claim that there is  $k \in \{\pm 1\}^{[i,j]}$  such that  $C_{[i,j]} \subseteq \ker \chi_k$  and  $k(i) = -1 = k(j)$ .

To see this, first note that since the index  $i$  was chosen as large as possible there is  $k_1 \in \{\pm 1\}^{[i,j]}$  with  $C_{[i,j]} \subseteq \ker \chi_{k_1}$  and  $k_1(i) = -1$ . Since  $j$  was chosen as small as possible, there is  $k_2 \in \{\pm 1\}^{[i,j]}$  with  $C_{[i,j]} \subseteq \ker \chi_{k_2}$  and  $k_2(j) = -1$ . If either  $k_1(j) = -1$  or  $k_2(i) = -1$ , then the claim is justified. Otherwise,  $k_1 k_2$  bears it out. Now construct  $\phi_k$  using such a  $k$ .

Since  $C_{[i,j]} \subseteq \ker \chi_k$ , the map  $T: \mathbb{C}[H, C] \rightarrow \mathbb{C}[H_{<i}, K_{<i}] \otimes \mathbb{C}[H_{>j}, K_{>j}]$  constructed from  $\phi_k$  is a surjective homomorphism. It remains to show that  $T(m_C^{(l)}) = 0$  if  $l \neq 0$ .

The element  $m_C^{(l)}$  maps to  $m_{K_{<i}} \otimes \mu \otimes m_{K_{>j}}$  under the isomorphism in (4), where  $\mu$  is the Haar measure of the restriction of the group  $C + \sigma^l(C)$  to  $[i, j]$ . To determine  $T(m_C^{(l)})$  it remains to calculate  $\phi_k(\mu)$ .

If  $l > 0$ , then  $\sigma^l(C)$  contains the element  $h_i$  of  $H$  defined by

$$h_i(n) = \begin{cases} (0, \bar{1}) & \text{if } n = i \\ (0, \bar{0}) & \text{otherwise} \end{cases}$$

and hence  $\mu$  is invariant under translation by  $h_i$ . Since  $k(i) = -1$ , the character  $\chi_k$  changes sign under translation by  $h_i$  and therefore  $\phi_k(\mu) = 0$ . Similarly, if  $l < 0$  then  $\phi_k(\mu) = 0$  because  $k(j) = -1$ .

We therefore have  $\phi_k(\mu) = 0$  for  $l \neq 0$  and hence  $T(m_C^{(l)}) = 0$  as required. □

In the next result we extend  $T$  to  $\mathbb{C}[G, C]$ . It will follow that for  $C$  a proper, compact, open subgroup of  $K$ ,  $\mathbb{C}[G, C]$  has quotients isomorphic to  $\mathbb{C}[D, F]$ .

LEMMA 3.5. *Let  $C$  be a proper open subgroup of  $K$ . Then, for some interval  $[i, j] \subseteq \mathbb{Z}$ , there is a surjective  $*$ -homomorphism*

$$\tilde{T}: \mathbb{C}[G, C] \rightarrow \mathbb{C}[H_{<i}, K_{<i}] \otimes \mathbb{C}[H_{>j}, K_{>j}].$$

Moreover the algebra  $\mathbb{C}[G, C]$  has quotients isomorphic to  $\mathbb{C}[D, F]$ .

*Proof.* We begin by proving the first statement. For each  $l \in \mathbb{Z}$  put

$$A_l := \text{span}\{m_C * \sigma^l h * m_C : h \in H\}.$$

Then  $A_0 = \mathbb{C}[H, C]$  and we obtain a grading  $\mathbb{C}[G, C] = \bigoplus_{l \in \mathbb{Z}} A_l$  with  $A_{l_1} * A_{l_2} \subseteq A_{l_1+l_2}$ . The identity

$$(m_C * \sigma^{-l} h_1 * m_C) * (m_C * \sigma^l h_2 * m_C) = m_C * \sigma^l(h_1) * m_{\sigma^l(C)} * h_2 * m_C$$

implies that, when  $l \neq 0$ ,  $A_{-l} * A_l$  is contained in the kernel of the homomorphism  $T$  defined in Lemma 3.4. The required homomorphism can therefore be obtained by setting  $\tilde{T}|_{A_0} = T$  and  $\tilde{T}|_{A_l} = 0$  for  $l \neq 0$ .

To see the second statement, use that  $\mathbb{C}[H_{<i}, K_{<i}] \otimes \mathbb{C}[H_{>j}, K_{>j}]$  is isomorphic to  $\mathbb{C}[H, K] \otimes \mathbb{C}[H, K] \cong \mathbb{C}[H, K]$ , hence has quotients isomorphic to  $\mathbb{C}[D, F]$ .  $\square$

We are now ready for the proof of our main result.

*Proof of Theorem 3.1.* Let  $C$  be a compact open subgroup of  $G$ . Any such subgroup is contained in  $H$ . The group  $K$  is a maximal compact subgroup of  $H$  and up to an inner automorphism of  $H$  all maximal compact subgroups of  $H$  are equal to  $K$ . Hence, up to an automorphism of  $G$  all maximal compact subgroups of  $G$  are equal to  $K$ . We may therefore assume from the outset that  $C$  is contained in  $K$ .

If  $C = K$ , then, the element  $\Lambda_K := m_K * h_0 * m_K$  satisfies the requirement of Theorem 3.1, as remarked after Lemma 3.3. If  $C \neq K$ , then  $\mathbb{C}[G, C]$  has a quotient isomorphic to  $\mathbb{C}[D, F]$  by Lemma 3.5. Any element mapping to  $\Lambda_1$  in  $\mathbb{C}[D, F]$  under the map  $\mathbb{C}[G, C] \rightarrow \mathbb{C}[D, F]$  satisfies the requirement for  $\Lambda_C$  in Theorem 3.1. The proof is complete.  $\square$

**4. A Conjecture.** The main ingredients in the construction of the example are that the group  $G$  has a compact open subgroup that is isomorphic to the product over the integers of copies of a finite group and that there is an inner automorphism of  $G$  which acts as the shift on this subgroup. Two conditions on a group  $G$  precluding such a structure have been studied in [1] and [2]. It may be that these conditions are the right ones to impose on  $G$  in order to guarantee that there is a compact open subgroup  $C$  such that  $\mathbb{C}[G, C]$  has a largest  $C^*$ -norm.

First, the *contraction group* of an automorphism  $\alpha$  is

$$\{x \in G \mid \alpha^k(x) \rightarrow e_G \text{ as } k \rightarrow \infty\}.$$

Contraction groups are shown in [1] to be closely related to the compact open subgroups tidy for  $\alpha$ . In particular, there are arbitrarily small subgroups tidy for  $\alpha$  if and only if the contraction group for  $\alpha$  is closed. The automorphism that is a shift on a product over the integers of copies of a finite group does not have closed contraction groups.

Second, for compact open subgroups  $U$  and  $V$  of  $G$  define

$$d(U, V) = \log([U : U \cap V][V : U \cap V]).$$

Then  $d$  is a *metric* on the compact open subgroups of  $G$  which plays an important role in the definition of the notion of *direction* of an automorphism [2]. A condition which seems to be important for the study of the space of directions of a group  $G$  is that balls in the metric space of compact open subgroups of  $G$  should be finite, that is, this metric space is proper. The infinite product of copies of a finite group does not satisfy this condition, which, however is satisfied if  $C$  is topologically finitely generated.

Ergodic abelian groups of automorphisms of totally disconnected locally compact groups are studied in [3] and in that paper a condition of ‘local finite generation’ is imposed. This condition implies the finiteness of balls condition and so may also be relevant to Hecke algebras.

CONJECTURE 4.1. *Let  $G$  be a totally disconnected locally compact group and suppose that  $G$  satisfies one of the following conditions:*

1. *For each compact open subgroup  $C$  and any real number  $R \geq 0$  the set  $\{C' \leq G \mid d(C, C') \leq R\}$  is finite;*
2. *Contraction groups of inner automorphisms of  $G$  are closed.*

*Then the Hecke algebra  $\mathbb{C}[G, C]$  has a largest  $C^*$ -norm whenever the compact open subgroup  $C$  is sufficiently small.*

Conditions 1 and 2 are independent because the full automorphism group of a homogeneous tree satisfies 1 but not 2, while the infinite product of copies of a finite group satisfies 2 but not 1. Both conditions are satisfied if  $G$  is a  $p$ -adic Lie group. At the time of writing it is still an open question whether the conjecture holds even in this case.

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