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HECKE ALGEBRAS OF GROUP EXTENSIONS

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We describe the Hecke algebra $\mathscr{H}(\Gamma, \Gamma_0)$ of a Hecke pair (Γ, Γ_0) in terms of the Hecke pair (N, Γ_0) where N is a normal subgroup of Γ containing Γ_0. To do this, we introduce twisted crossed products of unital $*$-algebras by semigroups. Then, provided a certain semigroup $S \subset \Gamma/N$ satisfies $S^{-1}S = \Gamma/N$, we show that $\mathscr{H}(\Gamma, \Gamma_0)$ is the twisted crossed product of $\mathscr{H}(N, \Gamma_0)$ by S. This generalizes a recent theorem of Laca and Larsen about Hecke algebras of semidirect products.

Key Words: Hecke algebras; Representation; Twisted crossed product by semigroups.

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1. INTRODUCTION

Suppose Γ is a group and Γ_0 is a subgroup of Γ. The pair (Γ, Γ_0) is called a *Hecke pair* if each double coset of Γ_0 in Γ is a finite union of left cosets; we also say that Γ_0 is a Hecke subgroup Γ. Normal subgroups, finite subgroups and subgroups of finite index are all Hecke subgroups, as are compact open subgroups of topological groups. The Hecke algebra $\mathscr{H}(\Gamma, \Gamma_0)$ is a convolution $*$-algebra of functions on the double coset space $\Gamma_0\backslash\Gamma/\Gamma_0$, which coincides with the group algebra $\mathbb{C}(\Gamma/\Gamma_0)$, when Γ_0 is normal.

Bost and Connes (1995) used the Hecke algebra of a particular Hecke pair $(P_{\mathbb{Q}}^{+}, P_{\mathbb{Z}}^{+})$ in their investigations of phase transitions in number theory. Laca and Raeburn (1999) showed that $\mathscr{H}(P_{\mathbb{Q}}^{+}, P_{\mathbb{Z}}^{+})$ could be realised as a crossed product of the group $*$-algebra $\mathbb{C}(\mathbb{Q}/\mathbb{Z})$ by a semigroup of endomorphisms, and this realisation proved useful in subsequent extensions of the analysis of Bost and Connes by Laca (1998) and Neshveyev (2002). Laca and Larsen (2003) subsequently extended the isomorphism of Laca and Raeburn (1999) to a broad class of Hecke algebras $\mathscr{H}(\Gamma, \Gamma_0)$ in which Γ is a semidirect product $N \rtimes G$ and Γ_0 is a subgroup of N. The results of Laca and Larsen (2003) are expressed using a family of $*$-algebraic crossed

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products by endomorphic actions of semigroups modelled on the C^*-algebraic crossed products used in analysis.

On reading Laca and Larsen (2003), we wondered whether it is necessary that the extension $e \to N \to \Gamma \to G \to e$ be split. One knows from operator algebras that results for semidirect products are often valid for nonsplit extensions, provided one is willing to twist the multiplication using cocycles (see Packer and Raeburn, 1989, for example). Here we introduce a family of twisted crossed products of $*$-algebras by endomorphic actions of semigroups motivated by constructions familiar to operator algebraists, and prove the following extension of the main theorem of Laca and Larsen (2003):

Theorem 1.1. *Let N be a normal subgroup of a group Γ, and let Γ_0 be a subgroup of N such that (Γ, Γ_0) is a Hecke pair. Let $\Sigma := \{\sigma \in \Gamma : \Gamma_0 \subset \sigma\Gamma_0\sigma^{-1}\}$ and let S be the image of Σ in Γ/N. Assume that $S^{-1}S = \Gamma/N$. Then there is a twisted action (α, u) of S on $\mathscr{H}(N, \Gamma_0)$ such that $\mathscr{H}(\Gamma, \Gamma_0)$ is isomorphic to $\mathscr{H}(N, \Gamma_0) \rtimes_{\alpha,u} S$.*

To make this article accessible to both analysts and algebraists, we introduce notation and include arguments which might be hard to dig out of the other camp's literature.

2. BASICS OF *-ALGEBRAS

All algebras in this article are complex associative algebras with a multiplicative identity denoted by 1. A $*$-algebra is an algebra $\mathscr{A}$ with a conjugate linear involution $a \mapsto a^*$ satisfying $(ab)^* = b^*a^*$. (It then follows that $1^* = 1$, for example). We call a^* the *adjoint* of a. By convention, all homomorphisms of $*$-algebras will be $*$-preserving, but not necessarily unital. Indeed, our most important examples are definitely not unital.

For analysts, the main example of a $*$-algebra is the algebra $B(H)$ of all bounded linear operators on a Hilbert space H, in which the involution is characterised by the equation

$$(Th \mid k) = (h \mid T^*k) \qquad \text{for } h, k \in H \quad \text{and} \quad T \in B(H).$$

An algebraist might prefer to think of a group $*$-algebra $\mathbb{C}G$, in which the adjoint of the element δ_g is by definition $\delta_{g^{-1}}$ for each group element $g \in G$. We can reconcile these points of view by observing that the group $*$-algebra is universal for unitary representations of G on Hilbert space.

An element v of a $*$-algebra is called a *unitary* if $v^*v = vv^* = 1$, an *isometry* if $v^*v = 1$, a *projection* if $v^2 = v = v^*$, and a *partial isometry* if $vv^*v = v$. The reason for this terminology is that whenever $\pi: \mathscr{A} \to B(H)$ is a representation of a $*$-algebra $\mathscr{A}$ as bounded operators on a Hilbert space H, $\pi(v)$ then has the appropriate geometric property. Thus, for example, if v is a projection in $\mathscr{A}$, then $\pi(v)$ is the orthogonal projection onto the closed subspace $\pi(v)H$, and if v is a partial isometry in $\mathscr{A}$ then $\pi(v)$ is a partial isometry in the usual sense that $\pi(v)$ is an isometry of $(\ker \pi(v))^{\perp}$ onto $\pi(v)H$.

When $v \in \mathscr{A}$ is a partial isometry, the elements v^*v and vv^* are projections, called the *initial* and *final* projections of v; for $v \in B(H)$, v is an isometry of $v^*vH = (\ker v)^{\perp}$ onto $vv^*H = vH$. In $B(H)$ and many other $*$-algebras, if vv^* or v^*v is a projection then v is a partial isometry (Berberian and Baer, 1972, Section 1.2), but this is not true in general (Berberian and Baer, 1972, Example 5A).

3. BASICS OF HECKE ALGEBRAS

Let (Γ, Γ_0) be a Hecke pair. The Hecke algebra $\mathscr{H}(\Gamma, \Gamma_0)$ of the pair (Γ, Γ_0) is the set of functions $f : \Gamma_0\backslash\Gamma/\Gamma_0 \to \mathbb{C}$ of finite support with multiplication

$$(fg)(\Gamma_0 x\Gamma_0) := \sum_{\Gamma_0 y\in\Gamma_0\backslash\Gamma} f(\Gamma_0 xy^{-1}\Gamma_0)\, g(\Gamma_0 y\Gamma_0) = \sum_{z\Gamma_0\in\Gamma/\Gamma_0} f(\Gamma_0 z\Gamma_0)\, g(\Gamma_0 z^{-1}x\Gamma_0) \tag{3.1}$$

and involution $f^*(\Gamma_0 x\Gamma_0) := \overline{f(\Gamma_0 x^{-1}\Gamma_0)}$. It is not obvious, but easy to check, that the formulas in (3.1) are independent of the choice of representative x for the double coset $\Gamma_0 x\Gamma_0$. The algebra $\mathscr{H}(\Gamma, \Gamma_0)$ is spanned by the characteristic functions $[x]$ of the double cosets $\Gamma_0 x\Gamma_0$ for $x \in \Gamma$. Indeed,

$$\big\{[x] : \Gamma_0 x\Gamma_0 \in \Gamma_0\backslash\Gamma/\Gamma_0\big\}$$

is a basis for $\mathscr{H}(\Gamma, \Gamma_0)$; the crucial point here is that we avoid duplication of elements by indexing over the double cosets of Γ_0 in Γ rather than the elements of Γ.

The next result is Corollary I.4.5 of Krieg (1990), but because it is essential to our arguments, we include a short proof. For $x \in \Gamma$, we denote by $L(x) := |\Gamma_0 x\Gamma_0/\Gamma_0|$ the number of left Γ_0-cosets in $\Gamma_0 x\Gamma_0$ and by $R(x) := L(x^{-1}) = |\Gamma_0\backslash\Gamma_0 x\Gamma_0|$ the number of right Γ_0-cosets in $\Gamma_0 x\Gamma_0$.

Lemma 3.1. *There is an algebra homomorphism* R: $\mathscr{H}(\Gamma, \Gamma_0) \to \mathbb{C}$ *such that* $R([x]) = R(x)$.

Proof. Since $\{[x] : \Gamma_0 x\Gamma_0 \in \Gamma_0\backslash\Gamma/\Gamma_0\}$ is a basis for $\mathscr{H}(\Gamma, \Gamma_0)$, the map: $[x] \to R(x)$ extends to a linear map R: $\mathscr{H}(\Gamma, \Gamma_0) \to \mathbb{C}$. We need to show that $R([x][y]) = R(x) R(y)$ for all $x, y \in \Gamma$. Let $x, y \in \Gamma$. Then

$$\begin{aligned}([x]\,[y])(\Gamma_0\delta\Gamma_0) &= \sum_{\Gamma_0 z\in\Gamma_0\backslash\Gamma} [x](\Gamma_0\delta z^{-1}\Gamma_0)[y](\Gamma_0 z\Gamma_0) = \sum_{\Gamma_0 z\subset\Gamma_0 y\Gamma_0} [x](\Gamma_0\delta z^{-1}\Gamma_0)\\ &= \big|\big\{\Gamma_0 z \in \Gamma_0\backslash\Gamma\colon \Gamma_0 z \subset \Gamma_0 y\Gamma_0, \Gamma_0\delta z^{-1} \subset \Gamma_0 x\Gamma_0\big\}\big|.\end{aligned}$$

Define $S_\delta = \{\Gamma_0 z \in \Gamma_0\backslash\Gamma : \Gamma_0 z \subset \Gamma_0 y\Gamma_0, \Gamma_0\delta z^{-1} \subset \Gamma_0 x\Gamma_0\}$, and note that as a consequence of the previous calculation we have $|S_\delta| = |S_{\delta'}|$ for all $\delta' \in \Gamma_0\delta\Gamma_0$.

Now choose a set of right coset representatives of Γ_0 in Γ, let c: $\Gamma_0\backslash\Gamma \to \Gamma$ be the corresponding section, and consider the set

$$T_\delta = \big\{(\Gamma_0 z, \Gamma_0 w) \in (\Gamma_0\backslash\Gamma) \times (\Gamma_0\backslash\Gamma)\colon \Gamma_0 z \subset \Gamma_0 y\Gamma_0, \Gamma_0 w \subset \Gamma_0 x\Gamma_0, \Gamma_0 wc(\Gamma_0 z) = \Gamma_0\delta\big\}.$$

Since $(\Gamma_0 z, \Gamma_0\delta z^{-1}) \in T_\delta$ for all $\Gamma_0 z \in S_\delta$ we have $|S_\delta| \le |T_\delta|$. The use of a section in the definition of T_δ ensures the second coordinate of each pair in T_δ is uniquely determined by the first coordinate, so that $|T_\delta| \le |S_\delta|$. Hence $|T_\delta| = |S_\delta|$; it follows that $|T_\delta| = |T_{\delta'}|$ for all $\delta' \in \Gamma_0\delta\Gamma_0$. Moreover,

$$[x][y] = \sum_{\Gamma_0\delta\Gamma_0 \in \Gamma_0\backslash\Gamma/\Gamma_0} |T_\delta|[\delta], \qquad \text{whence } R([x][y]) = \sum_{\Gamma_0\delta\Gamma_0 \in \Gamma_0\backslash\Gamma/\Gamma_0} |T_\delta| R(\delta)$$

by the linearity of R.

Let $\delta \in \Gamma$, and let $\{\delta_i\colon i = 1, \ldots, R(\delta)\}$ be a set of coset representatives for the right cosets in $\Gamma_0\delta\Gamma_0$, so that

$$\Gamma_0\delta\Gamma_0 = \bigsqcup_{i=1}^{R(\delta)} \Gamma_0\delta_i.$$

We have $|T_{\delta_i}| = |T_\delta|$ for each $i \in \{1, \ldots, R(\delta)\}$, and $T_{\delta_i} \cap T_{\delta_j} = \emptyset$ for $i \neq j$, so

$$|T_\delta|R(\delta) = \left|\bigsqcup_{i=1}^{R(\delta)} T_{\delta_i}\right| = \big|\{(\Gamma_0 z, \Gamma_0 w) \in (\Gamma_0\backslash\Gamma)^2\colon \Gamma_0 z \subset \Gamma_0 y \Gamma_0,$$
$$\Gamma_0 w \subset \Gamma_0 x \Gamma_0, \Gamma_0 w c(\Gamma_0 z) = \Gamma_0\delta\Gamma_0\}\big|.$$

Thus

$$\sum_{\Gamma_0\delta\Gamma_0 \in \Gamma_0\backslash\Gamma/\Gamma_0} |T_\delta|R(\delta) = \big|\{(\Gamma_0 z, \Gamma_0 w) \in (\Gamma_0\backslash\Gamma)^2\colon \Gamma_0 z \subset \Gamma_0 y\Gamma_0, \Gamma_0 w \subset \Gamma_0 x \Gamma_0\}\big|$$
$$= R(y)R(x),$$

so $R([x][y]) = R(x)R(y)$, as required. □

Remark 3.2. In general R is not a $*$-algebra homomorphism, essentially because $R(x)$ and $L(x) = R(x^{-1})$ may differ.

Corollary 3.3. *If x and y are elements of Γ with the property that $[x][y]$ is supported in the single double coset $\Gamma_0 xy\Gamma_0$, then*

$$[x][y] = \frac{R(x)R(y)}{R(xy)}[xy].$$

Proof. We know that $[x][y] = c[xy]$ is a multiple of $[xy]$; applying R to both sides gives $R(x)R(y) = cR(xy)$. □

We now set $\Sigma := \{\sigma \in \Gamma\colon L(\sigma) = 1\}$, and note that an element $\sigma \in \Gamma$ belongs to Σ if and only if $\Gamma_0\sigma\Gamma_0 = \sigma\Gamma_0$.

Corollary 3.4 (Brenken, 1999, Theorem 1.4)**.** *The map $W\colon \Sigma \to \mathscr{H}(\Gamma, \Gamma_0)$, defined by $W_\sigma = R(\sigma)^{-1/2}[\sigma]$ is a representation of the semigroup Σ by isometries.*

Proof. Since $W_\sigma^* = R(\sigma)^{-1/2}[\sigma^{-1}]$ and $\sigma \in \Sigma$ implies $\Gamma_0\sigma^{-1}\Gamma_0\sigma\Gamma_0 = \Gamma_0\sigma^{-1}\sigma\Gamma_0 = \Gamma_0$, Corollary 3.3 implies that

$$W_\sigma^* W_\sigma = R(\sigma)^{-1}[\sigma^{-1}][\sigma] = R(\sigma)^{-1}\frac{R(\sigma^{-1})R(\sigma)}{R(e)}[e],$$

which collapses to $[e]$ because $R(\sigma^{-1}) = L(\sigma) = 1$ for $\sigma \in \Sigma$. □

4. TWISTED CROSSED PRODUCTS BY SEMIGROUPS

Definition 4.1. A *twisted action* of a semigroup S on a unital $*$-algebra $\mathcal{A}$ is a pair (α, u) of maps $\alpha: S \to \operatorname{End}\mathcal{A}$ and $u: S \times S \to \mathcal{A}$ such that, for all $r, s, t \in S$ and $a \in \mathcal{A}$,

(1) $\alpha_e = \mathrm{id}_{\mathcal{A}}$,
(2) $u(s, e)^2 = u(s, e)^* = u(s, e) = u(e, s)$,
(3) $u(r, s)u(rs, t) = \alpha_r(u(s, t))u(r, st)$,
(4) $\alpha_{st}(a) = u(s, t)^*\alpha_s(\alpha_t(a))u(s, t)$,
(5) $(\alpha_s(\alpha_t(a)) = u(s, t)\alpha_{st}(a)u(s, t)^*$.

Definition 4.2. Given a twisted action (α, u) of a semigroup S on an unital $*$-algebra $\mathcal{A}$, the *twisted semigroup crossed product* $\mathcal{A} \rtimes_{\alpha,u} S$ is the unital $*$-algebra generated by the set $\{a: a \in \mathcal{A}\} \cup \{\mu_s: s \in S\}$ subject to the relations

$$\text{relations in } \mathcal{A}, \qquad \mu_s^*\mu_s = 1 = \mu_e, \quad \mu_s\mu_t = u(s, t)\mu_{st}, \quad \alpha_s(a) = \mu_s a \mu_s^*$$

for $s, t \in S$ and $a \in \mathcal{A}$. (Notice that formulas (1)–(5) of Definition 4.1 describe relations which hold in $\mathcal{A}$, and so also in $\mathcal{A} \rtimes_{\alpha,u} S$.)

Definition 4.3. Let $\mathcal{B}$ be a unital $*$-algebra. Given a twisted action (α, u) of a semigroup S on an unital $*$-algebra $\mathcal{A}$, a *covariant representation* of $(\mathcal{A}, S, \alpha, u)$ in $\mathcal{B}$ is a pair (π, U), where $\pi: \mathcal{A} \to \mathcal{B}$ is a unital homomorphism and $U: S \to \mathcal{B}$ is a map such that

$$U_s^*U_s = 1 = U_e, \qquad U_sU_t = \pi(u(s, t))\,U_{st}, \qquad \pi(\alpha_s(a)) = U_s\pi(a)U_s^*$$

for $s, t \in S$ and $a \in \mathcal{A}$. A covariant representation (π, U) of $(\mathcal{A}, S, \alpha, u)$ induces a unital homomorphism $\pi \times U : \mathcal{A} \rtimes_{\alpha,u} S \to \mathcal{B}$ called *the integrated form* of (π, U).

Anyone who is familiar with twisted crossed products by groups will have noticed that we have not imposed any structural properties on the values of our cocycles u. It turns out that we can make the values of the cocycles partial isometries without changing the crossed product.

Proposition 4.4. *Suppose (α, u) is a twisted action of a semigroup S on a unital $*$-algebra $\mathcal{A}$. Define w: $S \times S \to \mathcal{A}$ by*

$$w(s, t) = \alpha_s(\alpha_t(1))u(s, t)\alpha_{st}(1).$$

Then for all $s, t \in S$, $w(s, t)$ is a partial isometry with initial projection $\alpha_{st}(1)$ and final projection $\alpha_s(\alpha_t(1))$; the pair (α, w) is a twisted action of S on $\mathcal{A}$; and a representation (π, U) is covariant for $(\mathcal{A}, S, \alpha, u)$ if and only if it is covariant for $(\mathcal{A}, S, \alpha, w)$.

Proof. We first show that $w(s, t)$ is a partial isometry with initial projection $\alpha_{st}(1)$ and final projection $\alpha_s(\alpha_t(1))$. From parts (4) and (5) of Definition 4.1, we have

$$\begin{aligned}\alpha_s(\alpha_t(a))u(s, t) &= \alpha_s(\alpha_t(1))\alpha_s(\alpha_t(a))u(s, t)\\ &= u(s, t)\alpha_{st}(1)u(s, t)^*\alpha_s(\alpha_t(a))u(s, t)\\ &= u(s, t)\alpha_{st}(1)\alpha_{st}(a),\end{aligned}$$

from which we conclude that

$$\alpha_s(\alpha_t(a))u(s, t) = u(s, t)\alpha_{st}(a). \tag{4.1}$$

Using this equation with $a = 1$ we see that $w(s, t)$ can be written in one of three forms:

$$w(s, t) = \alpha_s(\alpha_t(1))u(s, t)\alpha_{st}(1) \tag{4.2}$$

$$= u(s, t)\alpha_{st}(1) \tag{4.3}$$

$$= \alpha_s(\alpha_t(1))u(s, t). \tag{4.4}$$

From (4.2), (4.4), and Definition 4.1(5) we obtain

$$w(s, t)w(s, t)^* = \alpha_s(\alpha_t(1))u(s, t)\alpha_{st}(1)u(s, t)^*\alpha_s(\alpha_t(1)) = \alpha_s(\alpha_t(1))^3 = \alpha_s(\alpha_t(1)).$$

This and (4.3) give

$$w(s, t)w(s, t)^*w(s, t) = \alpha_s(\alpha_t(1))u(s, t)\alpha_{st}(1) = w(s, t),$$

establishing that $w(s, t)$ is a partial isometry with final projection $\alpha_s(\alpha_t(1))$. Using (4.3), (4.2), and Definition 4.1(4), we calculate the initial projection of $w(s, t)$ to be

$$w(s, t)^*w(s, t) = \alpha_{st}(1)u(s, t)^*\alpha_s(\alpha_t(1))u(s, t)\alpha_{st}(1) = \alpha_{st}(1)^3 = \alpha_{st}(1).$$

Next we show that (α, w) is a twisted action of S on $\mathcal{A}$. It is straightforward to check formula (2) of Definition 4.1 for w. To obtain formula (3) for w, we compute using (4.4), (4.3), and formula (3) for u:

$$\begin{aligned}\alpha_r(w(s, t))w(r, st) &= \alpha_r(\alpha_s(\alpha_t(1))u(s, t))u(r, st)\alpha_{rst}(1)\\ &= \alpha_r(\alpha_s(\alpha_t(1)))\alpha_r(u(s, t))u(r, st)\alpha_{rst}(1)\\ &= \alpha_r(\alpha_s(\alpha_t(1)))u(r, s)u(rs, t)\alpha_{rst}(1),\end{aligned}$$

which by (4.3) is $\alpha_r(\alpha_s(\alpha_t(1)))u(r, s)w(rs, t)$. Thus

$$\begin{aligned}\alpha_r(w(s, t))w(r, st) &= \alpha_r(\alpha_s(1\alpha_t(1)))u(r, s)w(rs, t)\\ &= \alpha_r(\alpha_s(1))\alpha_r(\alpha_s(\alpha_t(1)))u(r, s)w(rs, t)\\ &= \alpha_r(\alpha_s(1))u(r, s)\alpha_{rs}(\alpha_t(1))w(rs, t),\end{aligned}$$

where we used (4.1) at the last step. Finally, from (4.4) we obtain

$$\begin{aligned}\alpha_r(w(s, t))w(r, st) &= \alpha_r(\alpha_s(1))u(r, s)\alpha_{rs}(\alpha_t(1))\alpha_{rs}(\alpha_t(1))u(rs, t)\\ &= \alpha_r(\alpha_s(1))u(r, s)\alpha_{rs}(\alpha_t(1))u(rs, t) = w(r, s)w(rs, t),\end{aligned}$$

which establishes formula (3) for w. Formula (4) for w follows from the corresponding formula for u and (4.4):

$$\begin{aligned}w(s, t)^*\alpha_s(\alpha_t(a))w(s, t) &= u(s, t)^*\alpha_s(\alpha_t(1))\alpha_s(\alpha_t(a))\alpha_s(\alpha_t(1))u(s, t)\\ &= u(s, t)^*\alpha_s(\alpha_t(a))u(s, t) = \alpha_{st}(a).\end{aligned}$$

Similarly, formula (5) for w follows from the corresponding formula for u and (4.3):

$$\begin{aligned}w(s, t)\alpha_{st}(a)w(s, t)^* &= u(s, t)\alpha_{st}(1)\alpha_{st}(a)\alpha_{st}(1)u(s, t)^*\\ &= u(s, t)\alpha_{st}(a)u(s, t)^* = \alpha_s(\alpha_t(a)).\end{aligned}$$

Thus (α, w) is a twisted action of S on $\mathcal{A}$.

To see that a representation (π, U) is covariant for $(\mathcal{A}, S, \alpha, u)$ if and only if it is covariant for $(\mathcal{A}, S, \alpha, w)$, we need only note that

$$\begin{aligned}\pi(w(s, t))U_{st} &= \pi(u(s, t)\alpha_{st}(1))U_{st} = \pi(u(s, t))\pi(\alpha_{st}(1))U_{st}\\ &= \pi(u(s, t))U_{st}\pi(1)U_{st}^*U_{st} = \pi(u(s, t))U_{st}U_{st}^*U_{st} = \pi(u(s, t))U_{st},\end{aligned}$$

so that the conditions of Definition 4.3 are satisfied with respect to u if and only if they are satisfied with respect to w. □

5. THE GROUP EXTENSION PROBLEM

Throughout this section, (Γ, Γ_0) will be a Hecke pair and N a normal subgroup of Γ such that $\Gamma_0 \subset N$. We write $\Sigma = \{\sigma \in \Gamma\colon L(\sigma) = 1\}$, and denote by W the isometric representation of Σ of Corollary 3.4. We denote by S the image of Σ under the quotient map $q\colon \Gamma \to \Gamma/N$, and we consider a fixed partial section $c\colon S \to \Sigma$ for q such that $c(e) = e$. The remaining ingredient in our standard notation is the homomorphism ι described in the following lemma.

Lemma 5.1. *Extending functions on* $\Gamma_0\backslash N/\Gamma_0$ *to be zero off* N *gives an injective unital homomorphism* $\iota\colon \mathcal{H}(N, \Gamma_0) \to \mathcal{H}(\Gamma, \Gamma_0)$.

Proof. Because $\Gamma_0 \subset N$, the convolution of two functions on $\Gamma_0\backslash\Gamma/\Gamma_0$ with support in $\Gamma_0\backslash N/\Gamma_0$ also has support in $\Gamma_0\backslash N/\Gamma_0$. □

The injection ι carries the characteristic function of $\Gamma_0 n\Gamma_0$ in $\mathcal{H}(N, \Gamma_0)$ to the characteristic function of $\Gamma_0 n\Gamma_0$ in $\mathcal{H}(\Gamma, \Gamma_0)$. It seems safe to use $[n]$ to denote either characteristic function, so that $\iota([n]) = [n]$.

Proposition 5.2. *There is an action β of Σ by endomorphisms of $\mathcal{H}(N, \Gamma_0)$ such that for $\sigma \in \Sigma$ and $n \in N$*

$$\beta_\sigma([n]) = \frac{1}{R(\sigma)} \sum_{\{[m]:\, m\in N, [\sigma^{-1}m\sigma]=[n]\}} [m], \tag{5.1}$$

and (ι, W) is then a covariant representation of $(\mathcal{H}(N, \Gamma_0), \Sigma, \beta)$ in $\mathcal{H}(\Gamma, \Gamma_0)$, in the sense that $\iota(\beta_\sigma([n])) = W_\sigma \iota([n])W_\sigma^*$.

Our proof of Proposition 5.2 shows simultaneously that there is such an action β and that (ι, W) is covariant, as does the proof of the corresponding result in Laca and Larsen (2003). The key is the following computation in $\mathcal{H}(\Gamma, \Gamma_0)$.

Lemma 5.3. *For each $n \in N$ and $\sigma \in \Sigma$*

$$W_\sigma[n]W_\sigma^* = \frac{1}{R(\sigma)} \sum_{[x]:\, [\sigma^{-1}x\sigma]=[n]} [x]. \tag{5.2}$$

Proof. We have $W_\sigma[n]W_\sigma^* = \frac{1}{R(\sigma)}[\sigma][n][\sigma^{-1}]$. Since $\sigma \in \Sigma$, $\Gamma_0\sigma\Gamma_0$ contains the single left coset $\sigma\Gamma_0$, and hence the sum in

$$[\sigma][n](\Gamma_0 x\Gamma_0) = \sum_{w\Gamma_0\in\Gamma/\Gamma_0} [\sigma](\Gamma_0 w\Gamma_0)[n](\Gamma_0 w^{-1}x\Gamma_0) = \sum_{w\Gamma_0\subset\Gamma_0\sigma\Gamma_0} [n](\Gamma_0 w^{-1}x\Gamma_0)$$

collapses to $[n](\Gamma_0\sigma^{-1}x\Gamma_0)$. Using the right-coset formula for the other convolution gives

$$\big(([\sigma][n])[\sigma^{-1}]\big)(\Gamma_0 x\Gamma_0) = \sum_{\Gamma_0 z\in\Gamma_0\backslash\Gamma} [n](\Gamma_0\sigma^{-1}xz^{-1}\Gamma_0)[\sigma^{-1}](\Gamma_0 z\Gamma_0),$$

and this sum similarly collapses to

$$[n](\Gamma_0\sigma^{-1}x\sigma\Gamma_0) = \begin{cases} 1 & \text{if } [\sigma^{-1}x\sigma] = [n] \\ 0 & \text{otherwise.} \end{cases}$$

But this is precisely $R(\sigma)$ times the right-hand side of (5.2). □

Proof of Proposition 5.2. Since W is a representation of Σ by isometries in $\mathcal{H}(\Gamma, \Gamma_0)$, $\operatorname{Ad} W$ is an action of Σ by endomorphisms of $\mathcal{H}(\Gamma, \Gamma_0)$. Because $\Gamma_0 \subset N$ and N is normal, every $[x] \in \mathcal{H}(\Gamma, \Gamma_0)$ with $[\sigma^{-1}x\sigma] = [n]$ has $x \in N$. Hence the right-hand side of (5.2) belongs to the range of ι, and indeed is the image under ι of the right-hand side of (5.1). Thus Lemma 5.3 implies: first, that $\operatorname{Ad} W_\sigma$ leaves the subalgebra $\iota(\mathscr{H}(N, \Gamma_0))$ invariant; second, that the resulting endomorphisms

$$\beta_\sigma := \iota^{-1} \circ \left(\operatorname{Ad} W_\sigma|_{\iota(\mathscr{H}(N,\Gamma_0))} \right) \circ \iota$$

satisfy (5.1); and, third, that the pair (ι, W) is covariant. □

Proposition 5.4. *There is a twisted action (α, u) of S on $\mathscr{H}(N, \Gamma_0)$ such that $\alpha = \beta \circ c$ and $\iota(u(s,t)) = W_{c(s)} W_{c(t)} W^*_{c(st)}$, and the pair $(\iota, W \circ c)$ is a covariant representation of $(\mathscr{H}(N, \Gamma_0), S, \alpha, u)$ in $\mathscr{H}(\Gamma, \Gamma_0)$. Moreover, each $u(s,t)$ is a partial isometry.*

Proof. To see that $W_{c(s)} W_{c(t)} W^*_{c(st)}$ belongs to the range of ι, we need to show that it has support in N. Since c takes values in Σ, we have $\Gamma_0 c(r) \Gamma_0 = c(r)\Gamma_0$ for every $r \in S$, and hence

$$\Gamma_0 c(s) \Gamma_0 c(t) \Gamma_0 c(st)^{-1} \Gamma_0 = c(s)c(t)\Gamma_0 c(st)^{-1} = \left(c(s)c(t)c(st)^{-1}\right) c(st)\Gamma_0 c(st)^{-1} \subset N.$$

Since ι is injective, it follows that u is well-defined.

It remains to verify the 5 conditions of Definition 4.1. (1) holds because $\alpha_e = \beta_{c(e)} = \beta_e$ is the identity. (2) also follows from $c(e) = e$. For (3), we compute:

$$\begin{aligned}
\iota(u(r,s)u(rs,t)) &= W_{c(r)} W_{c(s)} W^*_{c(rs)} W_{c(rs)} W_{c(t)} W^*_{c(rst)} \\
&= W_{c(r)} W_{c(s)} W_{c(t)} W^*_{c(st)} W^*_{c(r)} W_{c(r)} W_{c(st)} W^*_{c(rst)} \\
&= W_{c(r)} \iota(u(s,t)) W^*_{c(r)} \iota(u(r,st)) \\
&= \iota(\alpha_r(u(s,t)) u(r,st)),
\end{aligned}$$

where at the last step we used the covariance of (ι, W) for β. Property (5) also follows from the covariance of (ι, W):

$$\begin{aligned}
\iota(\alpha_s(\alpha_t(a))) &= W_{c(s)} W_{c(t)} \iota(a)\, W^*_{c(t)} W^*_{c(s)} \\
&= \iota(u(s,t))\, W_{c(st)}\, \iota(a)\, W^*_{c(st)} \iota(u(s,t)^*) \\
&= \iota(u(s,t)\alpha_{st}(a)u(s,t)^*). \qquad (5.3)
\end{aligned}$$

For (4), we first observe that

$$\begin{aligned}
\iota(u(s,t)^* u(s,t)) &= W_{c(st)} W^*_{c(t)} W^*_{c(s)} W_{c(s)} W_{c(t)} W^*_{c(st)} \\
&= W_{c(st)} W^*_{c(st)} \\
&= \iota(\beta_{c(st)}(1)) = \iota(\alpha_{st}(1)),
\end{aligned}$$

which, since ι is injective, implies that

$$u(s,t)^* u(s,t) = \alpha_{st}(1). \qquad (5.4)$$

Thus conjugating property (5) by $u(s, t)^*$ gives

$$u(s,t)^*\alpha_s(\alpha_t(a))u(s,t) = u(s,t)^*u(s,t)\alpha_{st}(a)u(s,t)^*u(s,t) = \alpha_{st}(1)\alpha_{st}(a)\alpha_{st}(1) = \alpha_{st}(a),$$

which is property (4) of Definition 4.1. The last claim follows from Corollary 3.4. □

Using the notation of this section, we can now give a more precise formulation of our main result, Theorem 1.1.

Theorem 5.5. *Suppose that N is a normal subgroup of Γ and that Γ_0 is a subgroup of N such that (Γ, Γ_0) is a Hecke pair. Adopt the standard notation of this section, and assume that $S^{-1}S = \Gamma/N$. Then $\imath \times (W \circ c)$ is an isomorphism of $\mathcal{H}(N, \Gamma_0) \rtimes_{\alpha,u} S$ onto $\mathcal{H}(\Gamma, \Gamma_0)$.*

Our strategy for proving Theorem 5.5 is to identify a spanning set for $\mathcal{H}(N, \Gamma_0) \rtimes_{\alpha,u} S$ which $\imath \times (W \circ c)$ carries into a basis for $\mathcal{H}(\Gamma, \Gamma_0)$.

Lemma 5.6. *We have*

$$\mathcal{H}(N, \Gamma_0) \rtimes_{\alpha,u} S = \operatorname{span}\{\mu_t^*[n]\mu_s : s, t \in S, n \in N\}. \tag{5.5}$$

Proof. Since the characteristic functions $[n]$ span $\mathcal{H}(N, \Gamma_0)$, the right-hand side of (5.5) contains all the generators of $\mathcal{H}(N, \Gamma_0) \rtimes_{\alpha,u} S$ and is closed under taking adjoints. Thus it suffices to prove that the right-hand side of (5.5) is closed under multiplication. So we let $s, t, p, q \in S$ and $a, b \in \mathcal{H}(N, \Gamma_0)$, and aim to prove that the product $(\mu_t^* a\mu_s)(\mu_q^* b\mu_p)$ belongs to the right-hand side of (5.5).

Since $S^{-1}S = \Gamma/N$, there exist $x, y \in S$ satisfying $sq^{-1} = x^{-1}y$, and hence also $xs = yq$. Then

$$\begin{aligned}(\mu_t^* a\mu_s)(\mu_q^* b\mu_p) &= \mu_t^*\mu_x^*\mu_x a\mu_x^*\mu_x\mu_s\mu_q^* b\mu_p \\ &= \mu_{xt}^* u(x,t)^*\alpha_x(a)u(x,s)\mu_{xs}\mu_q^* b\mu_p.\end{aligned}$$

By Proposition 5.4 the values of the cocycle u are partial isometries. Hence, by formula (5.4) and the relations in the crossed product, $u(y,q)^*u(y,q) = \alpha_{yq}(1) = \mu_{yq}\mu_{yq}^*$. We therefore have $u(y,q)^*u(y,q)\mu_{yq} = \mu_{yq} = \mu_{xs}$, and

$$\begin{aligned}(\mu_t^* a\mu_s)(\mu_q^* b\mu_p) &= \mu_{xt}^* u(x,t)^*\alpha_x(a)u(x,s)u(y,q)^*u(y,q)\mu_{yq}(\mu_q^* b\mu_p) \\ &= \mu_{xt}^* u(x,t)^*\alpha_x(a)u(x,s)u(y,q)^*\mu_y\mu_q(\mu_q^* b\mu_p) \\ &= \mu_{xt}^* u(x,t)^*\alpha_x(a)u(x,s)u(y,q)^*\alpha_y(\alpha_q(1)b)\mu_y\mu_p \\ &= \mu_{xt}^* u(x,t)^*\alpha_x(a)u(x,s)u(y,q)^*\alpha_y(\alpha_q(1)b)u(y,p)\mu_{yp}.\end{aligned}$$

But $u(x,t)^*\alpha_x(a)u(x,s)u(y,q)^*\alpha_y(\alpha_q(1)b)u(y,p)$ belongs to

$$\mathcal{H}(N, \Gamma_0) = \operatorname{span}\{[n] : n \in N\},$$

so $(\mu_t^* a\mu_s)(\mu_q^* b\mu_p)$ belongs to the right-hand side of (5.5), as required. □

We now consider the elements $\imath \times (W \circ c)\,(\mu_t^*[n]\mu_s) = W_{c(t)}^*[n]W_{c(s)}$ of $\mathscr{H}(\Gamma, \Gamma_0)$.

Lemma 5.7. *For all $\sigma, \tau \in \Sigma$ and $n \in N$,*

$$W_\tau^*[n]W_\sigma = \sqrt{\frac{R(\sigma)}{R(\tau)}}\frac{R(n)}{R(\tau^{-1}n\sigma)}[\tau^{-1}n\sigma]. \tag{5.6}$$

Proof. The support of $W_\tau^*[n]W_\sigma$ is $\Gamma_0\tau^{-1}\Gamma_0 n\Gamma_0\sigma\Gamma_0$; since τ and σ are in Σ, this is the single coset $\Gamma_0\tau^{-1}n\sigma\Gamma_0$. Applying Corollary 3.3 and remembering that $R(\tau^{-1}) = L(\tau) = 1$ gives the coefficient. □

The characteristic functions $[c(t)^{-1}nc(s)]$ are linearly independent, but many different choices of s, n, and t give the same characteristic function. Lemma 5.7 tells us that the spanning elements $\mu_t^*[n]\mu_s$ corresponding to these different choices may map under $\imath \times (W \circ c)$ to different scalar multiples of the same characteristic function. The next lemma makes these relationships explicit.

Lemma 5.8. *Suppose $s, t, p, q \in S$, $m, n \in N$ and $[c(t)^{-1}nc(s)] = [c(q)^{-1}mc(p)]$. Then*

$$\sqrt{\frac{R(c(t))}{R(c(s))}}\frac{1}{R(n)}\mu_t^*[n]\mu_s = \sqrt{\frac{R(c(q))}{R(c(p))}}\frac{1}{R(m)}\mu_q^*[m]\mu_p.$$

Proof. In view of Lemma 5.7, the hypothesis $[c(t)^{-1}nc(s)] = [c(q)^{-1}mc(p)]$ is equivalent to

$$\sqrt{\frac{R(c(t))}{R(c(s))}}\frac{1}{R(n)}W_{c(t)}^*\imath([n])W_{c(s)} = \sqrt{\frac{R(c(q))}{R(c(p))}}\frac{1}{R(m)}W_{c(q)}^*\imath([m])W_{c(p)}, \tag{5.7}$$

and we need to show that the corresponding relation holds in the crossed product.

Suppose that $[c(t)^{-1}nc(s)] = [c(q)^{-1}mc(p)]$. Since $\Gamma_0 \subset N$ this means

$$c(t)^{-1}nc(s)N = c(q)^{-1}mc(p)N,$$

and hence

$$t^{-1}s = q(c(t)^{-1}nc(s)N) = q(c(q)^{-1}mc(p)N) = q^{-1}p,$$

which is equivalent to $sp^{-1} = tq^{-1}$. Since $S^{-1}S = \Gamma/N$, there exist $a, b \in S$ such that $a^{-1}b = sp^{-1} = tq^{-1}$. Define $y = bp = as$ and $x = bq = at$. Computing in $\mathscr{H}(\Gamma, \Gamma_0)$ we obtain

$$\begin{aligned} W_{c(t)}^*\imath([n])W_{c(s)} &= W_{c(t)}^*W_{c(a)}^*W_{c(a)}\imath([n])W_{c(a)}^*W_{c(a)}W_{c(s)} \\ &= W_{c(at)}^*\imath(u(a,t))^*\imath(\alpha_a([n]))\imath(u(a,s))W_{c(as)}, \end{aligned}$$

and similarly

$$W_{c(q)}^*\imath([m])W_{c(p)} = W_{c(bq)}^*\imath(u(b,q))^*\imath(\alpha_b([m]))\imath(u(b,p))W_{c(bp)}.$$

Thus premultiplying (5.7) by $W_{c(x)}$ and postmultiplying by $W^*_{c(y)}$ gives

$$\sqrt{\frac{R(c(t))}{R(c(s))}}\frac{1}{R(n)}W_{c(x)}W^*_{c(x)}\,\iota\big(u(a,t)^*\alpha_a([n])u(a,s)\big)W_{c(y)}W^*_{c(y)}$$
$$=\sqrt{\frac{R(c(q))}{R(c(p))}}\frac{1}{R(m)}W_{c(x)}W^*_{c(x)}\iota\big(u(b,q)^*\alpha_b([m])u(b,p)\big)W_{c(y)}W^*_{c(y)},$$

This is equivalent to

$$\sqrt{\frac{R(c(t))}{R(c(s))}}\frac{1}{R(n)}\iota\big(\alpha_x(1)u(a,t)^*\alpha_a([n])u(a,s)\alpha_y(1)\big)$$
$$=\sqrt{\frac{R(c(q))}{R(c(p))}}\frac{1}{R(m)}\iota\big(\alpha_x(1)u(b,q)^*\alpha_b([m])u(b,p)\alpha_y(1)\big). \tag{5.8}$$

Since ι is injective, (5.8) implies that

$$\sqrt{\frac{R(c(t))}{R(c(s))}}\frac{1}{R(n)}\alpha_x(1)\,u(a,t)^*\alpha_a([n])\,u(a,s)\alpha_y(1)$$
$$=\sqrt{\frac{R(c(q))}{R(c(p))}}\frac{1}{R(m)}\alpha_x(1)\,u(b,q)^*\alpha_b([m])\,u(b,p)\alpha_y(1)$$

in $\mathscr{H}(N,\Gamma_0)$. In the crossed product $\mathscr{H}(N,\Gamma_0)\rtimes_{\alpha,u}S$, we can rewrite this as

$$\sqrt{\frac{R(c(t))}{R(c(s))}}\frac{1}{R(n)}\mu_x\mu_x^*\,u(a,t)^*\mu_a[n]\mu_a^*u(a,s)\mu_y\mu_y^*$$
$$=\sqrt{\frac{R(c(q))}{R(c(p))}}\frac{1}{R(m)}\mu_x\mu_x^*u(b,q)^*\mu_b[m]\mu_b^*u(b,p)\mu_y\mu_y^*.$$

Premultiplying both sides of this expression by μ_x^* and postmultiplying by μ_y removes the outside terms μ_x and μ_y^*. Now we can undo the calculations we did above, this time working in the crossed product, to arrive at

$$\sqrt{\frac{R(c(t))}{R(c(s))}}\frac{1}{R(n)}\mu_t^*[n]\mu_s=\sqrt{\frac{R(c(q))}{R(c(p))}}\frac{1}{R(m)}\mu_q^*[m]\mu_p,$$

which is the desired conclusion. □

Proof of Theorem 5.5. Let $\gamma\in\Gamma$. Since $S^{-1}S=\Gamma/N$ there exist $s,t\in S$ such that $\gamma N=t^{-1}s$, and then $\gamma N=c(t)^{-1}c(s)N=c(t)^{-1}Nc(s)$. Thus γ has the form $\gamma=c(t)^{-1}nc(s)$ for some $n\in N$, and we can define

$$b(\Gamma_0\gamma\Gamma_0)=\sqrt{\frac{R(c(t))}{R(c(s))}}\frac{R(c(t)^{-1}nc(s))}{R(n)}\mu_t^*[n]\mu_s;$$

Lemma 5.8 shows that $b(\Gamma_0\gamma\Gamma_0)$ is independent of the choice of s, t and n. Lemma 5.6 implies that the set

$$\{b(\Gamma_0\gamma\Gamma_0)\colon \Gamma_0\gamma\Gamma_0 \in \Gamma_0\backslash\Gamma/\Gamma_0\}$$

spans $\mathscr{H}(N, \Gamma_0) \rtimes_{\alpha,u} S$. Lemma 5.7 implies that

$$(\imath \times (W \circ c))(b(\Gamma_0\gamma\Gamma_0)) = [c(t)^{-1}nc(s)] = [\gamma],$$

and hence the image of the spanning set $\{b(\Gamma_0\gamma\Gamma_0)\colon \Gamma_0\gamma\Gamma_0 \in \Gamma_0\backslash\Gamma/\Gamma_0\}$ under $\imath \times (W \circ c)$ is a basis for $\mathscr{H}(\Gamma, \Gamma_0)$. It now follows from elementary linear algebra that the linear map $\imath \times (W \circ c)$ is bijective. □

Remark 5.9. The triples of groups (Γ_0, N, Γ) satisfying the hypotheses of Theorem 5.5 are closed under taking direct products, in the sense that if (Λ_0, M, Λ) is another such triple, then $(\Gamma_0 \times \Lambda_0, N \times M, \Gamma \times \Lambda)$ is another. Indeed, if $S \subset \Gamma/N$ and $T \subset \Lambda/M$ are the associated semigroups, then the corresponding semigroup for the direct product is $S \times T$. Since the extension

$$e \to N \times M \to \Gamma \times \Lambda \to \Gamma/N \times \Lambda/M \to e$$

splits if and only if both factors do, we can deduce from this observation and the examples provided in Laca and Larsen (2003) that there are lots of examples of nonsplit extensions in which all our ingredients are nontrivial.

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